Chapter 1

Electromagnetism

1.1 Electrical and Magnetic Forces

Forces due to charge arise because electric charges attract or repel by nature (so-called electrostatic forces, although the charges need not be motionless), and also when they move relative to one another, so-called electromagnetic or electrodynamic forces. The magnitude of the latter is typically weaker by order $1/c^2$ relative to the former.

1.1.1 Electrostatic Forces and Fields

Given two charges q_1 and q_2 in an isotropic, continuous medium, the empirical law due to Coulomb states

$$\mathbf{F}_{1\to 2} = k_E \frac{q_1 q_2}{r^2}\hat{\mathbf{r}} = k_E \frac{q_1 q_2}{r^3}\mathbf{r} \tag{1.1}$$

where k_E is a constant characteristic of the permittivity and the dielectric nature of the space in which the interaction occurs; its value and units depend on the operative system of measurement. When the sign of the charges is the same, the force is repulsive; if opposite, attractive. To be Lorentz invariant, the source charge must be at rest in the reference frame, though at velocities small with respect to c, $\mathbf{F}_{1\to 2} = -\mathbf{F}_{2\to 1}$.

The contribution of any single charge to the total electrostatic force on a second charge is independent of other charges, the resultant being a vector summation of all charges:

$$\mathbf{F}_{\text{total on q}} = \mathbf{F}_{1\to 1} + \mathbf{F}_{2\to q} + \mathbf{F}_{3\to q} + \cdots \tag{1.2}$$

An electrostatic field, indicating the force per unit charge, is associated with an electrostatic force. $\mathbf{E}$ is a vector in the direction of $\mathbf{F}$ on a unit positive charge q at its instantaneous location:

$$\mathbf{E} = \frac{1}{q}\mathbf{F} \tag{1.3}$$

or, equivalently, the force on a point charge q where $\mathbf{E}$ is the electric field is

$$\mathbf{F} = q\mathbf{E} \tag{1.4}$$

The electric field at any point due to a distribution of charges is found by summing or integrating vectorially the fields independently at the point:

$$\mathbf{E}_{\text{total}} = \mathbf{E}_1 + \mathbf{E}_2 + \mathbf{E}_3 + \cdots \tag{1.5}$$

Problems

1. Find the field of a uniformly charged sphere (charge density ρ) of radius R.
2. Show that inside a uniformly charged spherical shell the electric field due to the shell's charge is zero.

1.1.2 Force between Magnetic Poles

As he did for the force between two electric charges, Coulomb found the strength of the interaction between two magnetic poles is proportional to the strength of the poles, q_m, and inversely proportional to the square of their separation, r:

$$F_M = \frac{k_M}{2}\frac{q_{m1}q_{m2}}{r^2} \tag{1.6}$$

where k_m is a factor depending on the nature (permeability) of the medium in which the poles reside and the units employed.

1.1.3 Magnetic Field Intensity

The magnetic field intensity, B, at any point is the ratio of the magnetic pole force due to a pole Q_m to the magnitude of a (small) north pole:

$$B = \frac{F_M}{q_m} = \frac{k_M}{2}\frac{Q_m}{r^2} \tag{1.7}$$

This, like the electric field intensity, is a vector directed between the pole that is the source of the field and the (positive) test pole. Magnetic field lines can be mapped and drawn in a way similar to electric field lines, except that there is no such object as an isolated magnetic pole.

1.1.4 Magnetic Moment

The magnetic moment of a thin bar magnet (width $\ll$ length) is the product of the length, ℓ, and the pole strength, q_m:

$$M = q_m\ell \tag{1.8}$$

Placed in a uniform magnetic field, B, each pole experiences a force q_mB, leading to a torque around the center of the bar magnet [see figure 1.1]:

$$\tau = 2q_m\vec{B}\times\frac{\ell}{2}\hat{n} = q_mB\ell\sin\theta = MB\sin\theta \tag{1.9}$$

where $\hat{n}$ is the unit vector along the bar magnet and θ is the angle between this unit vector and the direction of the uniform magnetic field.

The work dW done in rotating the magnet $d\theta$ is

$$dW = MB\sin\theta d\theta \tag{1.10}$$

The work done by the external field to align the magnet with the field from an orientation perpendicular to the direction of the external field is

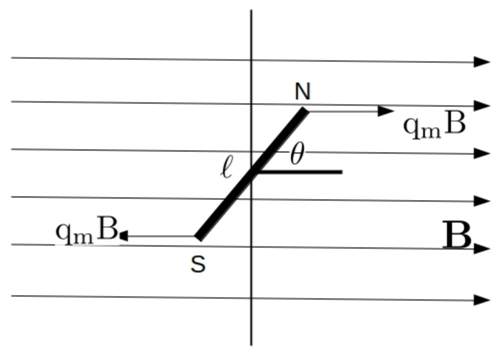

Figure 1.1: Torque on a bar magnet in a uniform magnetic field.

$$\begin{aligned} W &= -\int_{\frac{\pi}{2}}^{0} MB \sin\theta d\theta \\ &= MB \end{aligned} \tag{1.11}$$

Work is required to magnetize a substance.

1.1.5 Magnetic Field Sources

A charge located at $\mathbf{r}$ and moving with velocity $\mathbf{v}$ relative to an inertial coordinate system, is the source of a magnetic field

$$\mathbf{B} = \frac{k_M}{2}\frac{q}{r^2}(\mathbf{v}\times\mathbf{r}) \tag{1.12}$$

This implies that $B = 0$ along the line of motion but constant in circles centered around and perpendicular to the motion.

Since a current in a differential length of wire $d\mathbf{l}$ can be related to $d\mathbf{v}$, $Id\mathbf{l} = qdv$, the differential magnetic field due to the current element is

$$d\mathbf{B} = \frac{k_M}{2}\frac{I}{r^3}(d\mathbf{l}\times\mathbf{r}) \tag{1.13}$$

a relationship known as the Biot-Savart law. Integration yields $\mathbf{B}$ over the length (complete loop) of the wire.

Problem

Calculate the field on the axis of a circular current loop of radius a.

1.1.6 Magnetic Forces on Moving Charge

If a charged particle experiences an acceleration due to its motion relative to an inertial frame, then a magnetic field exists. The force inducing the acceleration is:

$$\mathbf{F} = q(\mathbf{u} \times \mathbf{B}), \tag{1.14}$$

where $\mathbf{u}$ is the particles velocity and $\mathbf{B}$ is the field vector called the magnetic field or the magnetic induction. Note that the converse of the existence proof is not true: even if a magnetic field is present, if $\mathbf{u} \parallel \mathbf{B}$, the particle will experience no force. Note, too, that the direction of the force is perpendicular to the motion, so no work is done (the force is non-conservative; there is no scalar potential associated with $\mathbf{B}$).

In the presence of both $\mathbf{E}$ and $\mathbf{B}$,

$$\mathbf{F} = q[\mathbf{E} + (\mathbf{u} \times \mathbf{B})] \tag{1.15}$$

a relationship called the Lorentz equation.

In the case of a current-carrying wire,

$$d\mathbf{F} = I(d\mathbf{s} \times \mathbf{B}), \tag{1.16}$$

where $d\mathbf{s}$ is the direction of the current at that point.

A flat coil lying in the $x - y$ plane in a uniform magnetic field will experience no net force, but will experience a torque

$$d\tau = \mathbf{r} \times d\mathbf{F}, \tag{1.17}$$

which, when integrated, gives

$$\tau = I\left[\left(B_y \oint y\ dx\right)\hat{\mathbf{i}} + \left(B_x \oint x\ dy\right)\hat{\mathbf{j}}\right] \tag{1.18}$$

But the area $A = \oint x\ dy = -\oint y\ dx$, so

$$\tau = IA(-B_y\hat{\mathbf{i}} + B_x\hat{\mathbf{j}}) = IA(\hat{\mathbf{k}} \times \mathbf{B}) \tag{1.19}$$

The coil will rotate until its plane is perpendicular to $\mathbf{B}$. More generally, if $\mathbf{A}$ is the vector representing the area of the coil, with a direction determined by the right-hand rule, following the current, then

$$\tau = I(\mathbf{A} \times \mathbf{B}) \tag{1.20}$$

If there are n turns to the coil, then the coefficient of the cross-product is nI.

The flux $d\Phi_B$ of $\mathbf{B}$ through a differential area $d\mathbf{A}$ is

$$d\Phi = \mathbf{B} \cdot d\mathbf{A} \tag{1.21}$$

so,

$$\Phi = \int_S \mathbf{B} \cdot d\mathbf{A} \tag{1.22}$$

But there are no magnetic charges, so field lines are continuous and close on themselves.

$$\Phi_B = 0 \quad \text{closed surface} \tag{1.23}$$

1.2 Flux and Gauss's Law

If Q represents the sum of charges distributed in a region of space, S represents an arbitrarily shaped surface of area A completely enclosing Q, $\mathbf{E}$ the electric field due to Q at some point P on S, and $\hat{\mathbf{n}}$ the normal vector from S at P, then

$$d\mathbf{A} = \hat{\mathbf{n}} dA \tag{1.24}$$

is the vector element of area at P and

$$d\Phi_E = \mathbf{E} \cdot d\mathbf{A} \tag{1.25}$$

is the electric flux through dA. If lines of force (directed curves tangent to $\mathbf{E}$ at all points) are drawn with normal density E, then the electric flux is the number of lines cutting dS.

The total flux (outward) through S:

$$\Phi_E = \oint_S \mathbf{E} \cdot d\mathbf{A} \tag{1.26}$$

Gauss's Law states that the total flux out of an arbitrary closed surface is proportional to the net electric charge enclosed by the surface:

$$\Phi_E = 4\pi k_E Q. \tag{1.27}$$

The Biot-Savart law can be rewritten as an integral (analogous to Coulomb's law and Gauss's law $\mathbf{F} = kq_E Q/r^3 \; d\hat{\mathbf{r}} \Leftrightarrow \oint_S \mathbf{E} \cdot d\mathbf{A} = 4\pi k_E Q$)

$$\oint_C \mathbf{B} \cdot d\mathbf{s} = 2\pi k_M I \tag{1.28}$$

called Ampère's circuital law. The integral is around an arbitrary closed path, while I is the total current through an open surface bounded by the path:

$$I = \int_S \mathbf{J} \cdot d\mathbf{A}, \tag{1.29}$$

where J is the current density. The orientation is determined by a right-hand rule where the thumb points along the current and the fingers wrap around the path.

1.3 Electric Potential

Static electric forces are conservative, so a test charge possess potential energy U in a static electric field numerically equal to the work done against the field in moving the charge from infinity to the point $(x,\ y,\ z)$:

$$U(x,\ y,\ z) = -\int_{\infty}^{(x,\ y,\ z)} \mathbf{F} \cdot d\mathbf{s} \tag{1.30a}$$

$$= -\int_{\infty}^{(x,\ y,\ z)} F_x dx + F_y dy + F_z dz \tag{1.30b}$$

Conversely,

$$\mathbf{F} = -\nabla U = -\left(\frac{\partial U}{\partial x}\mathbf{i} + \frac{\partial U}{\partial y}\mathbf{j} + \frac{\partial U}{\partial z}\mathbf{k}\right) \tag{1.31}$$

The electric potential ϕ, or voltage V, is the electric potential energy per unit test charge

$$\phi(x,\ y,\ z) = \frac{U(x,\ y,\ z)}{q}$$

$$= -\int_{\infty}^{(x,\ y,\ z)} \frac{\mathbf{F}}{q} \cdot d\mathbf{s} \tag{1.32a}$$

$$= -\int_{\infty}^{(x,\ y,\ z)} \mathbf{E} \cdot d\mathbf{s} \tag{1.32b}$$

$$= -\int_{\infty}^{(x,\ y,\ z)} E_x dx + E_y dy + E_z dz \tag{1.32c}$$

Conversely,

$$\mathbf{E} = -\nabla\phi = -\left(\frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}\right) \tag{1.33}$$

Near a point charge located at the origin,

$$\mathbf{E} = k_E \frac{q}{r^3}\mathbf{r} \tag{1.34}$$

The potential is found by integrating from infinity to $\mathbf{r} = (x,\ y,\ z)$ along a straight line to the origin, noting that $d\mathbf{r} = -d\mathbf{s}$ and that the electric field points out from the origin:

$$\phi = -\int_{\infty}^{r} \mathbf{E} \cdot d\mathbf{s} = -\int_{\infty}^{r} k_E \frac{q\ dr'}{r'^2} = k_E \frac{q}{r} \tag{1.35}$$

A surface (curve) over (along) which the potential is constant is called an equipotential surface (curve), described in an equation by

$$\phi(x,\ y,\ z) = c \tag{1.36}$$

where a surface is uniquely defined by a unique constant c. Note that,

$$d\phi = 0 = \mathbf{E} \cdot d\mathbf{s}, \tag{1.37}$$

implying that $\mathbf{E} \perp d\mathbf{s}$ for $d\mathbf{s}$ along the equipotential.

Since electric fields superpose vectorially, potentials add algebraically.

$$\phi_{\text{total}} = \phi_1 + \phi_2 + \phi_3 + \cdots \tag{1.38}$$

Since scalar addition is simpler than vector addition, it's usually simpler to find ϕ_{total} and then differentiate to find $\mathbf{E}_{\text{total}}$ than to do the vector addition. For a finite charge distribution of density ρ, then, the differential potential a distance r from a volume element dv of the distribution is

$$d\phi = k_E \frac{\rho\ dv}{r}, \tag{1.39}$$

so the total potential at some point $(x,\ y,\ z)$ is

$$\phi(x,\ y,\ z) = k_E \int_V \frac{\rho\ dv}{r}. \tag{1.40}$$

When the electric potential of a unit charged particle (e.g., electron or proton) decreases by 1 volt, the particle loses 1 eV of potential energy and gains 1 eV of kinetic energy, 1 eV$= 1.602 \times 10^{-19}$J.

1.4 Field Energy

We've seen that charging and magnetizing objects take work. There's more energy in the system after being charged or magnetized than before. In classical field theory, that energy is conceived as being stored in the field, and the quantity calculated is the energy per unit volume.

The capacitance of a parallel plate capacitor is

$$C = \frac{A}{4\pi k_E s} \tag{1.41}$$

where A is the area of the plates, s their separation, and, again, k_E includes the permittivity of the dielectric between them. Then,

$$W = \frac{1}{2}CV^2 = \frac{AE^2 s^2}{8\pi k_E s} = \frac{E^2}{8\pi k_E}(As) \tag{1.42}$$

or

$$w = \frac{W}{As} = \frac{E^2}{8\pi k_E} \tag{1.43}$$

In fact, this result is general. In the case of a non-uniform electric field, small volume units can be summed.

Similarly, the energy per unit volume in a magnetic field is

$$w = \frac{B^2}{4\pi k_M} \tag{1.44}$$

1.5 Maxwell's Equations

1.5.1 Gauss' Law for Electricity

The electric flux out of any closed surface is proportional to the total charge enclosed by the surface.

Integral form:

$$\oint_S \mathbf{E} \cdot d\mathbf{A} = 4\pi k_E Q \tag{1.45}$$

Differential form:

$$\nabla \cdot \mathbf{E} = 4\pi k_E \rho \tag{1.46}$$

The area integral of the electric field measures the net charge enclosed. The divergence of the electric field measures the density of sources and implies conservation of charge.

1.5.2 Gauss' Law for Magnetism

As in the case of electricity, the net magnetic flux through a closed surface amounts to a statement about the sources of magnetic field.

Integral form:

$$\oint_S \mathbf{B} \cdot d\mathbf{A} = 0 \tag{1.47}$$

Differential form:

$$\nabla \cdot \mathbf{B} = 0 \tag{1.48}$$

If there were a magnetic monopole source, the area integral would be finite. Because the divergence of a vector field is proportional to the point source density, Gauss' law for magnetic fields affirm that there are no magnetic monopoles.

1.5.3 Faraday's Law of Induction

A time-varying magnetic field is always accompanied by a spatially-varying electric field.

Integral form:

$$\oint_C \mathbf{E} \cdot d\mathbf{s} = -\frac{d\Phi_B}{dt} \tag{1.49}$$

Differential form:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{1.50}$$

The line integral $\oint_C \mathbf{E} \cdot d\mathbf{s} = \epsilon$, the emf directed so as to counteract the varying magnetic field, whose rate of change it is proportional to. This relation is the physical bases for inductors, transformers, and electric generators.

1.5.4 Modified Ampère's Law

Ampère's Ciruital Law states that the current in a loop of wire is proportional to the line integral of magnetic field enclosed by the loop as long as all electric fields are static. Maxwell, from symmetry and conservation considerations, suggested that a time-varying electric field is accompanied by a spatially-varying magnetic field.

Integral form:

$$\oint_C \mathbf{B} \cdot d\mathbf{s} = 2\pi k_M I + \frac{2k_E}{k_M} \frac{\partial}{\partial t} \int_S \mathbf{E} \cdot d\mathbf{A} \tag{1.51}$$

Differential form:

$$\nabla \times \mathbf{B} = 2\pi k_M \left(\mathbf{J} + \frac{1}{4\pi k_E} \frac{\partial \mathbf{E}}{\partial t} \right) \tag{1.52}$$

Note that $\frac{2k_E}{k_M} = c^2$, synthesizing electromagnetism with light, electromagnetic waves.

Problem

If $\mathbf{B}$ is a circularly collimated uniform but time-varying magnetic field:

$$\mathbf{B} = B_z \hat{k} = \begin{cases} (B_0 \cos \omega t)\hat{k} & r < r_0 \\ 0 & r > r_0 \end{cases}$$

What is $\mathbf{E} = E_\theta$, the electric field along some loop within $(r < r_0)$ and outside $(r > r_0)$ $\mathbf{B}$?

1.5.5 Charge Conservation

The current flowing out through the surface of a volume must equal the rate at which charge within the volume decreases:

$$\begin{aligned}
\oint_S \mathbf{J} \cdot d\mathbf{A} &= -\frac{d}{dt} \iiint \rho dV \\
\iiint \nabla \cdot \mathbf{J} dV &= -\iiint \frac{\partial \rho}{\partial t} dV \\
\nabla \cdot \mathbf{J} &= -\frac{\partial \rho}{\partial t} \qquad (1.53)
\end{aligned}$$

using the divergence theorem.

Taking the divergence of the modified Ampère's Law, using Gauss' Law for electricity, and knowing that the divergence of a curl is zero,

$$\begin{aligned}
\nabla \cdot \nabla \times \mathbf{B} &= 2\pi k_M \left(\nabla \cdot \mathbf{J} + \frac{1}{4\pi k_E} \frac{\partial \nabla \cdot \mathbf{E}}{\partial t} \right) \\
0 &= 2\pi k_M \left(\nabla \cdot \mathbf{J} + \frac{1}{4\pi k_E} 4\pi k_E \frac{\partial \rho}{\partial t} \right) \\
0 &= \nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} \qquad (1.54)
\end{aligned}$$