Special Relativity Review

0.1 Galilean Transformations

0.1.1 Events and Coordinates

A physical event happens at a position in space and at an instant in time. The position is assigned three position coordintates that measure its displacement from the designated origin of a coordinate system, and the instant is assigned a clock reading relative to an initial instant.

Two different observers, O and O', equipped with equivalent measuring sticks and clocks, assign different coordinates depending on their relative velocity v, even when they agree on origin ($x = x' = 0$) and initial instant ($t = t' = 0$).

0.1.2 Galilean Coordinate Transformations

$$x' = x - vt \quad y' = y \quad z' = z \quad t' = t \tag{1}$$

0.1.3 Galilean Velocity Transformations

$$u'_x = \frac{dx'}{dt'} = \frac{dx'}{dt} = u_x - v \quad u'_y = u_y \quad u'_z = u_z \tag{2}$$

0.1.4 Galilean Acceleration Transformations

$$a'_x = \frac{du'_x}{dt'} = \frac{du'_x}{dt} = \frac{du_x}{dt} = a_x \quad a'_y = a_y \quad a'_z = a_z \tag{3}$$

0.1.5 Invariance of an Equation

An equation is invariant if it has the same form when determined by different observers. Classically, the Galilean transformations relate different observers' space and time measurements. If the resulting form of an equation is the same, the equation is said to be invariant under the Galilean transformations.

Problems

1. Given a mass oscillating on a horizontal, frictionless surface at the end of a massless spring, show that, under Galilean transformations, the equations of motion are invariant with respect to an observer's motion along the direction of the oscillation.

2. Show that $c^2t^2-x^2-y^2-z^2$ and $c^2dt^2-dx^2-dy^2-dz^2$ are not invariant under Galilean transformations.

3. Show that the electromagnetic wave equation, which follows from Maxwell's electromagnetic equation, is not invariant under Galilean transformations.

4. Show that kinetic energy will be found to be conserved in one-dimensional elastic collisons by all inertial observers who move along the axis of the collision.

0.2 Postulates of Special Relativity

0.2.1 The Ether and Absolute Space

Light, being a wave (albeit electromagnetic), was felt to need a medium in which to propagate. The ether, permeating all space, was postulated.

While (classical) mechanics (Newton's Laws) is invariant under the Galilean transformation, electromagnetism as formulated by Maxwell is not. The observer who measures light traveling at velocity $c = 1/\sqrt{\epsilon_0\mu_0}$ is at rest relative to "absolute" space.

0.2.2 Ether Wind

As the earth moves in circles, an observer on it should, on the basis of classical mechanics and the Galilean transformations, be able to detect evidence of passing through the ether. Experiments designed to measure such an effect, most famously that by Michelson and Morley, could find no indication of this relative motion.

A Michelson-Morley interferometer may be oriented with either arm parallel to the "ether wind."

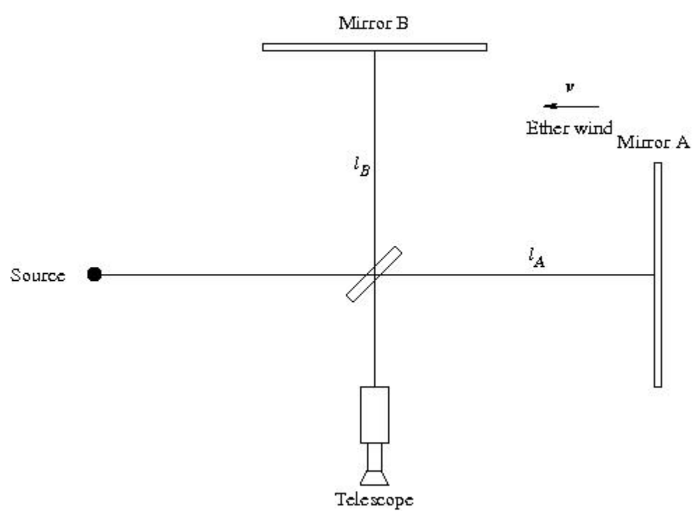

Figure 1: Michelson-Morley interferometer.

In the illustrated configuration, where the source emits along the line of motion, the return flight-time along arm A, using Galilean transformations, is

$$\Delta t_A^{\|} = \frac{l_A}{c-v} + \frac{l_A}{c+v} = \frac{2l_A/c}{1-(v^2/c^2)}. \tag{4}$$

For $v \ll c$,

$$\Delta t_A^{\|} \approx \frac{2l_A}{c}\left(1 + \frac{v^2}{c^2}\right). \tag{5}$$

Along arm B, the return flight-time is

$$\Delta t_B^{\|} = \frac{2l_B}{\sqrt{c^2-v^2}} = \frac{2l_B/c}{\sqrt{1-(v^2/c^2)}}, \tag{6}$$

and, for $v \ll c$,

$$\Delta t_B^{\|} \approx \frac{2l_B}{c}\left(1 + \frac{v^2}{2c^2}\right). \tag{7}$$

Thus, the difference in flight-times is

$$\delta^{\|} = \Delta t_A^{\|} - \Delta t_B^{\|} \approx \frac{2}{c}\left(l_A - l_B + \frac{v^2}{c^2}(l_A - \frac{l_B}{2})\right). \tag{8}$$

When the interfermoter is rotated by 90°, the source emits perpendicular to the motion, so

$$\Delta t_A^{\perp} = \frac{2l_A}{\sqrt{c^2-v^2}} = \frac{2l_A/c}{\sqrt{1-(v^2/c^2)}} \approx \frac{2l_A}{c}\left(1 + \frac{v^2}{2c^2}\right), \tag{9}$$

and

$$\Delta t_B^{\perp} = \frac{2l_B c}{c^2-v^2} = \frac{2l_B/c}{1-(v^2/c^2)} \approx \frac{2l_B}{c}\left(1 + \frac{v^2}{c^2}\right). \tag{10}$$

And the difference in flight-times is

$$\delta^{\perp} = \Delta t_A^{\perp} - \Delta t_B^{\perp} \approx \frac{2}{c}\left(l_A - l_B + \frac{v^2}{c^2}(\frac{l_A}{2} - l_B)\right). \tag{11}$$

The interference pattern will shift a number of fringes depending on the difference between these flight time differences relative to the period of the light emitted:

$$\Delta N = \frac{\delta^{\|} - \delta^{\perp}}{T} \approx \frac{c}{\lambda}\frac{2}{c}\frac{v^2}{c^2}\frac{l_A + l_B}{2} = \frac{v^2}{c^2}\frac{l_A + l_B}{\lambda}, \tag{12}$$

to order v^2/c^2.

Based on the principle that physical quantities must be defined operationally, Einstein questioned the validity of the Galilean transformations, particularly $t' = t$. Defining two equidistant events as simultaneous if light signals from each reach the same position at the same clock reading, it can be shown that such simultaneity for one observer will not be registered by another observer moving relative to the first.

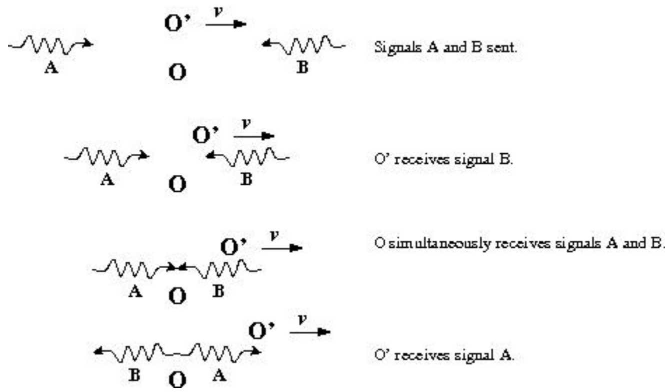

Figure 2: Sequence of equidistant events in two reference frames.

Problem

If O' registers the two events as simultaneous, how does O register the sequence?

0.2.3 Principle of Relativity

All observers in inertial frames should find the laws of physics are invariant. In particular, electromagnetic laws should be just as invariant as dynamic laws, and, thus, light speed should be measured as c by all initial observers regardless of the motion of the source.

Problem

How might a light pulse be used to synchronize a clock located a distance ℓ from an observer with a clock at the observer's position?

0.3 The Lorentz Coordinate Transformations

0.3.1 The Lorentz Coordinate Transformations

The Principle of Relativity requires that the Galilean transformations be replaced by the Lorentz transformations.

$$x' = \frac{x - vt}{\sqrt{1 - (v^2/c^2)}} \quad y' = y \quad z' = z \quad t' = \frac{t - \frac{v}{c^2}x}{\sqrt{1 - (v^2/c^2)}} \tag{13}$$

Inverting,

$$t = \sqrt{1 - (v^2/c^2)}t' + \frac{v}{c^2}x,$$

then

$$\begin{aligned}
x - v\left(\sqrt{1-(v^2/c^2)}t' + \frac{v}{c^2}x\right) &= x'\sqrt{1-(v^2/c^2)} \\
x[1-(v^2/c^2)] &= (x'+vt')\sqrt{1-(v^2/c^2)} \\
x &= \frac{x'+vt'}{\sqrt{1-(v^2/c^2)}} \\
t &= \sqrt{1-(v^2/c^2)}t' + \frac{v}{c^2}\frac{x'+vt'}{\sqrt{1-(v^2/c^2)}} \\
t &= \frac{t'+(v/c^2)x'}{\sqrt{1-(v^2/c^2)}}
\end{aligned}$$

$$x = \frac{x'+vt'}{\sqrt{1-(v^2/c^2)}} \quad y = y' \quad z = z' \quad t = \frac{t'+(v/c^2)x'}{\sqrt{1-(v^2/c^2)}} \tag{14}$$

0.3.2 Simultaneity

To an observer who records two events occurring at the same clock reading, these occur simultaneously. By the Galilean transformations, such a simultaneous occurrence for one observer would be simultaneous for all inertial observers. In a world abiding by the Relativity Principle, events that are simultaneous for one observer will not, in general, be simultaneous for other observers.

Say, $t_A = t_B$. Then

$$t'_B - t'_A = \frac{(t_B - t_A) - (v/c^2)(x_B - x_A)}{\sqrt{1-(v^2/c^2)}} = \frac{(v/c^2)(x_A - x_B)}{\sqrt{1-(v^2/c^2)}}, \tag{15}$$

which implies that only if $v = 0$ or $x_A = x_B$ will O' as well as O record events A and B to be simultaneous.

Realize that only one clock is sufficient to determine simultaneity if two events occur at the same position, but two, synchronized clocks are required to determine simultaneity if the events are displaced.

0.4 Measuring Lengths

Length is defined as the difference between the spatial coordinates of an object's endpoints. In general, these coordinates must be determined at the same time. However, the object is not moving relative to the observer, then the coordinate determinations may be made at any time, in which case the length is referred to as the object's rest or proper length.

If we designate the proper length of an object as L_0, then, if the object is at rest with respect to O', $L_0 = x'_B - x'_A$, and

$$x'_B - x'_A = \frac{(x_B - x_A) + v(t_B - t_A)}{\sqrt{1-(v^2/c^2)}}. \tag{16}$$

For O to measure the length of this object, which is seen by O to be moving, the endpoints must be determined simultaneously, $t_A = t_B$, so

$$L = x_B - x_A = (x'_B - x'_A)\sqrt{1-(v^2/c^2)} = L_0\sqrt{1-(v^2/c^2)}. \tag{17}$$

Because $\sqrt{1-(v^2/c^2)} < 1$, $L < L_0$, a result referred to as the Lorentz or Lorentz-Fitzgerald contraction.

0.5 Measuring Time

0.5.1 Proper Time

An observer who notes the interval between two events as determined with a single clock at the same position, $\Delta t_0 = t_B - t_A$ at $x_A = x_B$, measures the proper time interval between the two events.

0.5.2 Time Dilation

To an observer moving relative to that clock's position, so, to this observer, that clock is moving. To this observer, the events take place at different positions, $x'_A \neq x'_B$, and so measuring the time interval between them requires separate, synchronized clocks, $\Delta t' = t'_B - t'_A$.

$$\Delta t' = \frac{\Delta t_0 - (v/c^2)(x_B - x_A)}{\sqrt{1-(v^2/c^2)}}. \tag{18}$$

But, $x_A = x_B$.

$$\Delta t' = \frac{\Delta t_0}{\sqrt{1-(v^2/c^2)}}. \tag{19}$$

Since $\sqrt{1-(v^2/c^2)} < 1$, $\Delta t' > \Delta t_0$, the time interval of the single clock will be seen by the moving observer to be moving slower, that is, to be dilated.

But note that the relationship is symmetric, depending on the frame in which one clock is at rest. A single clock, when read by an observer who sees the clock moving, will tick slower than the clocks of this observer fixed at different locations.

0.6 Relativistic Space-Time Measurements

Problems

1. The equation for a spherical pulse of light starting from the origin at $t = t' = 0$ is

$$c^2t^2 - x^2 - y^2 - z^2 = 0.$$

   Show from the Lorentz transformations that O' will measure pulse to be spherical and consequently that light velocity is the same for all observers.

2. Show that the following differential expression is invariant under a Lorentz transformation:

$$c^2dt^2 - dx^2 - dy^2 - dz^2.$$

3. Show that the electromagnetic wave equation is invariant under a Lorentz transformation:

$$\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} - \frac{\partial^2\phi}{\partial x^2} - \frac{\partial^2\phi}{\partial y^2} - \frac{\partial^2\phi}{\partial z^2} = 0$$

0.7 Relativistic Velocity Transformations

Problems

1. From the Lorentz coordinate transformations, derive the Lorentz velocity transformations.
2. Derive the inverse Lorentz velocity transformations.

0.8 Momentum and Energy

0.8.1 Momentum

It was shown previously that the differential expression, $c^2dt^2 - dx^2 - dy^2 - dz^2$ is invariant, typically designated ds^2. For an observer, $dx = dy = dz = 0$ locally, so locally $ds^2 = c^2dt^2$. In this case, the time is the proper time, referred to earlier in a finite interval as Δt_0, but typically written in the infinitesimal case $d\tau$, so $ds^2 = c^2d\tau^2$. The relationship between this proper time and a measured time interval is

$$d\tau = dt\sqrt{1 - \frac{v^2}{c^2}} \tag{20}$$

Note that $d\tau^2 = ds^2/c^2$, and is therefore invariant.

It is convenient to define a velocity relative to $d\tau$, sometimes called proper velocity

$$\boldsymbol{\eta} = \frac{d\mathbf{x}}{d\tau} \tag{21}$$

related to a measured velocity, $\mathbf{v} = \frac{d\mathbf{x}}{dt}$, as

$$\boldsymbol{\eta} = \frac{\mathbf{v}}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{22}$$

The utility of proper velocity emerges in four-vector notation, but here it suffices to note that for momentum conservation to hold, even at relativistic speeds, two observers who move relative to one another must measure the same value for the momentum of an object, say, moving perpendicular to their relative motion. But the velocity transformations indicate that the two observers will not measure the same velocity $\mathbf{v}$ in that direction. Thus, momentum in its classical definition $m\mathbf{v}$ will not be the same in different inertial frames, but

$$\mathbf{p} = m\boldsymbol{\eta} = \frac{m\mathbf{v}}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{23}$$

allows momentum conservation to be consistently maintained in special relativity.

0.8.2 Energy

Relativistic energy is defined

$$E = \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{24}$$

Problems

1. Notice that in the classical limit, $v \ll c$, $\mathbf{p} \to m\mathbf{v}$, the classical momentum. The same is not true for relativistic energy. Expand Equation 24 in a Taylor series and show that

$$E = E_0 + K$$

where $E_0 = mc^2$ is the so-called rest energy and K is the relativistic kinetic energy. Show that K reduces to the classical kinetic energy in the same low velocity limit.

2. Show that $\frac{E^2}{c^2} - p^2$ is an invariant. What is the relationship between relativistic energy and momentum for a massless particle? What is a massless particle's velocity, v?

0.8.3 Newton's Second Law

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} = \frac{d}{dt}(m\boldsymbol{\eta}) = \frac{d}{dt}\left(\frac{m\mathbf{v}}{\sqrt{1-\frac{v^2}{c^2}}}\right). \tag{25}$$

0.8.4 The Mass-Energy Relationship

If the physics is to remain unchanged at high speeds, then the work-kinetic energy theorem must hold:

$$W = \int_{s_1}^{s_2} \mathbf{F} \cdot d\mathbf{s} = \Delta K. \tag{26}$$

Problems

1. Show that, for one-dimensional motion starting from rest, $\Delta K = E - E_0 = mc^2\left(\frac{1}{\sqrt{1-v^2/c^2}} - 1\right)$.

2. Use the binomial expansion to show that $\Delta K = E - E_0 \approx \frac{1}{2}mv^2$ when $v/c \ll 1$.

0.8.5 Relationship between Momentum and Energy

It was found previously that $\frac{E^2}{c^2} - p^2 = m^2c^2$.

Problems

1. Find the expression for Newton's second law, $F = \frac{dp}{dt}$, for the case in which the net force acts in the direction of the particle's velocity.

2. Two identical particles, each with rest mass number m_0, collide head-on inelastically with equal velocities v. What is the mass number, M, of the composite body, as measured in the laboratory?

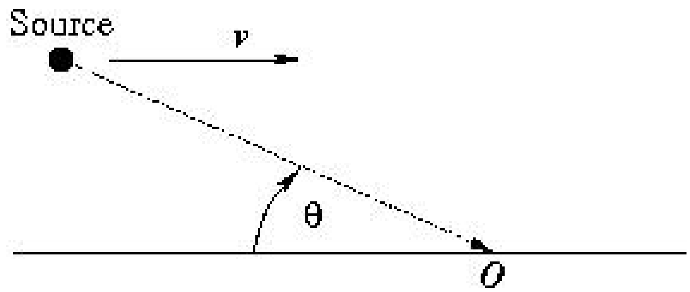

Figure 3: Radiation emitting source.

0.9 Doppler Effect

A source emits electromagnetic radiation of frequency ν_0 in its rest frame. It moves with velocity v at an angle θ relative to an observer.

The frequency measured by the observer is given by the Doppler equation:

$$\nu = \nu_0 \frac{\sqrt{1 - (v^2/c^2)}}{1 - (v/c)\cos\theta}. \tag{27}$$

As the velocity of the radiation is c in all frames, the corresponding wavelength shift is:

$$\lambda = \frac{c}{\nu}. \tag{28}$$

Problem

For the situation when the source and observer are directly receding from one another ($\theta = \pi$), evaluate the Doppler equation to first order in v/c.