Week 12: Four-Momentum (Undergraduate)

Four-Momentum

Notation: four-vectors are indicated with a bold capital letter; three-vectors are indicated with a lowercase letter under an arrow; vector components are indicated by italicized letters.

In special relativity, a four-vector (or 4-vector) is a four-component object that transforms via the Lorentz transformation. Space-time is one such four-vector. Four-momentum, $\boldsymbol{P}$ is the most important relativistic four-vector for particle and nuclear physics.

$$\boldsymbol{P} \equiv (E, \vec{p}) = (E, p_x, p_y, p_z)$$

where E is the relativistic energy, $E = \gamma mc^2 = \gamma m$ in natural dimensions, and $\vec{p} = (p_x, p_y, p_z)$ is the relativistic three-momentum, $p_x = \beta_x \gamma mc = \gamma m v_x$, etc. for p_y and p_z, where $\beta_x = v_x/c = v_x$, $\gamma = 1/\sqrt{1-\beta^2} = 1/\sqrt{1-|\vec{v}|^2}$, where, $|\vec{v}| = \sqrt{v_x^2 + v_y^2 + v_y^2}$. From this, $|\vec{p}| = \beta\gamma mc = \gamma m|\vec{v}|$. **Note that in this context, $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ refer to the motion of an object in the (inertial) frame in which the observer is at rest.**

The relativistic kinetic energy is

$$K = E - mc^2 = E - m,$$

where m is the mass of the object whose four-momentum is being referred to. Furthermore,

$$\frac{|\vec{p}|c}{E} = \beta \Rightarrow \frac{|\vec{p}|}{E} = |\vec{v}|.$$

Finally, the most useful relation for nuclear and particle physics, the magnitude of the four-vector, which yields the (invariant) mass of the object:

$$m^2c^4 = E^2 - |\vec{p}|^2c^2 \Rightarrow m^2 = E^2 - |\vec{p}|^2.$$

Notice, in particular, the minus sign in the calculation of this magnitude. The gauge [the (non-Euclidian) geometry] of special relativity dictates that the magnitude of a four-vector is the "time" component squared minus the square of the "space" vector components.

Objects of mass $m = 0$ move at $|\vec{v}| = c = 1$, and conversely, any object moving at $|\vec{v}| = 1$ has zero mass.[1] This implies that $E = |\vec{p}|$ for massless particles, like photons. With such a particle, all of its energy is kinetic.

The Lorentz tranformations transforms the four-momentum between inertial frames:

[1] $E = \gamma m$ is always finite. $\gamma \to \infty$ as $|\vec{v}| \to 0$, so m must be zero for E to be finite.

$$\begin{bmatrix} E' \\ p'_x \\ p'_y \\ p'_z \end{bmatrix} = \begin{bmatrix} \gamma(E - \beta p_x) \\ \gamma(p_x - \beta E) \\ p_y \\ p_z \end{bmatrix}$$

Note that here β and γ refer to the motion of the prime frame relative to the unprimed frame, oriented such that the x-axes of the two frames are parallel.

1. **What is the magnitude of the velocity of a particle whose total energy is 10% larger than its mass (rest energy)?**

2. **A particle of mass m moves in the $-y$ direction with a kinetic energy $K = \frac{2}{3}m$. In terms of m, what are the components of the particle's four-momentum?**

3. **Cosmic rays are very high energy subatomic particles, some of which have energies as large as several Joules. (How the particles obtain these energies remains something of an open question.)**

   (a) **A proton of total energy 1.5 J traverses the Milky Way (diameter about 10^5 lightyears). According to a clock traveling with the proton, how long would the trip take?**

   (b) **A photon starts the same journey at the same time as the proton. How far ahead of the proton is the photon when the latter reaches the other end of the galaxy? [Hint: When $\beta \approx 1$, $1 - \beta^2 \approx 2(1 - \beta)$]**

4. **Imagine an experiment in which a particle of mass m_0 transforms ("decays") into to two particles of mass m_1 and m_2. Before it decayed, m_0 was moving in the laboratory frame at $\beta_0 = \frac{3}{5}$ in the $+z$ direction. After the decay, m_1 moves in the $+z$ direction of the laboratory frame at $\beta_1 = \frac{4}{5}$, while m_2 emerges essentially at rest.**

   (a) **Show that $m_1 = \frac{3}{4}m_0$ and $m_2 = \frac{1}{4}m_0$ if total (particle) mass and Newtonian momentum are conserved in the laboratory frame.**

   (b) **Show that $m_1 = \frac{9}{16}m_0$ and $m_2 = \frac{5}{16}m_0$ if four-momentum is conserved in the laboratory frame. What happened to particle mass?**

   (c) **Show (with Einstein's velocity transformation equations) that in m_0's rest-frame, the z components of the three velocities are $v'_{0z} = 0$, $v'_{1z} = \frac{5}{13}$, and $v'_{2z} = -\frac{3}{5}$.**

   (d) **Given the masses found in part (4a) and the transformed velocities found in part (4c), show that Newtonian momentum is not conserved in m_0's rest-frame.**

(e) Show that, with the masses found in part (4b) and the transformed velocities found in part (4c), four-momentum is conserved in m_0's rest-frame.

5. A proton (mass $m_p = 0.938$ GeV $= 938$ MeV in natural units), initially at rest, is "struck" by a particle known as a pion (mass $m_\pi = 140$ MeV) moving in the $+x$ direction with momentum $|\vec{p}_{\pi,i}| = 900 \pm 40$ MeV. The particles scatter in the x-y plane with the proton moving at an angle $(20 \pm 1)^\circ$ relative to the $+x$ axis with three-momentum magnitude $|\vec{p}_{p,f}| = 1040 \pm 40$ MeV, while the pion (back-)scatters at an angle $(-109 \pm 1)^\circ$ relative to the $+x$ axis with three-momentum magnitude $|\vec{p}_{\pi,f}| = 390 \pm 40$ MeV.

(a) What are the components of the scattered particles' three-momentum, $p_{\pi_x,f}$, $p_{\pi_y,f}$, $p_{p_x,f}$, $p_{p_y,f}$?

(b) Considering uncertainties, are the measured values of the three-momenta consistent with classical (Newtonian) three-momentum conservation?

(c) Considering uncertainties, are the measured values of three-momentum consistent with an elastic scattering event in the classical (Newtonian) sense: (Newtonian) kinetic energy is conserved?

(d) Since no additional particles are created by the collision, and since the identities of the particles were not changed, and since quantum mechanics tells us that subatomic particles have no internal energy, the collision must be elastic. Does treating the momenta relativistically lead to conservation of energy?

Four-Momentum Conservation

The magnitude of an object's four-momentum is its mass, $m^2 = E^2 - |\vec{p}|^2$. This implies that the mass of a system of particles is **not** the sum of the particle masses, but the magnitude of the total four-momentum of the system, which is typically larger or smaller than the sum of the masses. Mass is a property of the system, and only some of it resides in the constituent particles. For example, the mass of a system at rest (that is, one in which the total momentum is zero) is the sum of the *energies* of the constituent particles, not the sum of their masses. Again, the mass M of a system of two identical particles, each of mass m, which collide totally inelastically (they stick together) is not $2m$ before or after the collision, but $M > 2m$ and unchanged by the collision. Although the total mass of a system is not the simple sum of its constituent masses, that mass is frame-independent (all interial frame observers will measure the same total mass, regardless of their frame).

6. Annihilation occurs when a particle collides with its corresponding antiparticle. Their energies are converted entirely into electromagnetic energy (photons). The best possible rocket engine would mix matter with an equal quantity of corresponding antimatter and collimate the resulting photons into a tight beam directed out the rear nozzle of the engine. Imagine such a rocket at rest (in some inertial frame) in deep space. The total mass of the rocket at rest, including both matter and antimatter fuels, is $M = 90,000$ kg. A firing of the engine results in photons with a total energy E being emitted to the rear and the rocket moving forward at $|\vec{v}| = v_x = \frac{4}{5}$. What is the rocket's mass, m, after the firing?

7. Because momentum is conserved, matter-antimatter annihilation typically results in the creation of two or more photons. If only two are created, they must move away from each other in opposite directions in the center-of-mass frame. In an instance of annihilation that creates two photons, one photon has energy E while the other has energy $4E$ as measured in the laboratory frame. What is the mass M of the system?

8. An electron with kinetic energy $K = m_e$ collides with a positron (antielectron) at rest. The subsequent annihilation produces two photons, one of which moves perpendicular to the electron's original trajectory. The other moves at an angle θ relative to the electron's original trajectory. What are the photons' energies (in terms of m_e) and the angle θ?

9. A particle of mass m at rest transforms ("decays") into two identical particles, each of which has mass $\frac{1}{3}m$. What is the (relativistic) kinetic energy of each particle, in terms of m? (Recall that total three-momentum must be conserved.)

10. The lightest elmentary particle containing a strange quark is called a kaon. Actually, there are four types of kaon, two neutral versions and two charged versions. The most stable of the four is dubbed the long-lived neutral kaon, or K-long, K_L^0. It has a mass $M = 498$ MeV and a half-life of approximately 36 ns (nanoseconds, or 10^{-9} s). About 1/1000 K_L^0s transforms ("decays") into two identical particles called pions. There are three types of pion, two charged and one neutral. The neutral pion, π^0, has mass $m = 135$ MeV. If a long-lived neutral kaon at rest decays to two neutral pions, what is the magnitude of the pions' velocity? (Recall, $|\vec{p}|/E = \beta$.) Note that 2×135 Mev < 498 MeV. What happened to the mass?

11. A negatively charged pion, π^-, has mass $m_\pi = 140$ MeV. It usually transforms ("decays") into a negatively charged muon,

μ^-, which has mass $m_\mu = 106$ MeV, and a(n anti-)neutrino, whose mass is negligible. If a π^- at rest transforms in this manner, what is the magnitude of the muon's velocity, β_μ?

12. Compton Scattering I: The process in which a photon scatters from a (quasi-free) electron at rest is known as Compton Scattering. Analyze the situation in which a photon of energy E_0 reflects exactly in the opposite direction of its original motion as a result of the scatter. That is, find an expression for the scattered photon's energy, E in terms of E_0 and m_e, the mass of the electron.

13. Compton Scattering II: Find the general expression for Compton scattering. That is, find the scattered photon's energy, E, after it scatters at an angle θ with respect to its original direction, again in terms of E_0 and m_e.