Week 1: "Natural" Dimensions (Undergraduate)

Particle, nuclear, and even atomic physicists often find it convenient to choose energy as the basic dimension $[E]$ and its unit the electron-volt. One electron-volt is the amount of kinetic energy an electron gains when accelerated across a potential difference of 1 volt. Typical energies in atomic physics range up to a few hundred electron volts. Nuclear energies range between 10^3 (kilo-) and 10^6 (mega-) electron-volts (keV - MeV). Particle physics studies energies from 10^6 (mega-) electron-volts and up. Most phenomena presently studied in particle physics are in the 10^9 (giga-) electron-volt range (GeV).

These physicists also tend to employ a system of dimensions and units which *define* the speed of light c, the reduced Planck's constant $\hbar \equiv \frac{h}{2\pi}$, the vacuum permittivity ε_0, and the vacuum permeability μ_0 to be dimensionless quantities equal to 1: $c = \hbar = \varepsilon_0 = \mu_0 \equiv 1$ (dimensionless). This system of dimensions and units is referred to as "natural." The physical constants c and h characterize Special Relativity and Quantum Mechanics, and energy is arguably the most important quantity computed and measured in particle and nuclear physics.

The theories of particle physics, in particular, but also, to some extent, of nuclear physics, are based on Quantum Field Theory, which is the merger of Special Relativity and Quantum Mechanics, and so c and/or h appear in most, if not all, of the most frequently used equations in these disciplines. Carrying them around proves burdensome. It also makes very little sense to use a system of dimension and units that describes the ordinary world, where velocities are referenced to 0 and energies are on the order of Jules. Not much work is necessary to make objects of very small mass go very fast. For example, 500 kV (the voltage in some U.S. transmission lines) will accelerate an electron to 10% of light speed, at which the electron has a kinetic energy of 8×10^{-14} J, which in natural units is 500 keV (kilo-electron-volts) and the velocity is 0.1, unitless. When things move at a significant fraction of the speed of light, it just makes sense to reference speeds to that of light, as a fraction, and therefore unitless.

To summarize, at least with regard to the treatment of c, because

1. the typical kinematics of elementary particles is relativistic,
2. zero velocity is arbitrary (reference frame-dependent), and
3. light speed, c, is measured to have the same magnitude in all reference frames,

designating light speed as the reference speed is a natural choice.

In general, nuclear and particle physicists report velocities as a fraction of c,

$$\beta \equiv v/c$$

which is obviously dimensionless. But because $c = 1$ and dimensionless, $\beta = v$, which is also dimensionless and has a value between 0 and 1.

What about other quantities? The distance dimension $[L]$ and the time dimension $[T]$ are related to velocity by

$$[v] = \frac{[L]}{[T]}$$

Since v is dimensionless, $[L]$ and $[T]$ must be the same natural dimension. To find it, recall that a photon has energy,

$$E = \hbar\omega = \frac{2\pi\hbar c}{\lambda}$$

where ω is angular velocity, which has dimension $1/[T]$; λ is wavelength, which has dimension $[L]$. From these,

$$[T] = [L] = \frac{1}{[E]}$$

that is, the natural dimension of both time and distance is inverse energy.[1]

Recall from Special Relativity that

$$p = \beta\gamma mc$$

is the relativistic momentum, and

$$E = \gamma mc^2$$

Thus, the ratio

$$\frac{pc}{E} = \beta = v$$

and c are both dimensionless, so momentum's natural dimension is also energy.

Because, also from Special Relativity,

$$M^2c^4 = E^2 - p^2c^2$$

the natural dimension of mass is energy, as well.

One electron-volt, again, is the amount of kinetic energy an electron gains when accelerated across a potential difference of 1 volt. Assuming, for simplicity, that this acceleration occurs in a capacitor,

$$\Delta K = |\vec{F}||\Delta\vec{s}| = q|\vec{E}||\Delta\vec{s}| = qV$$

where K refers to kinetic energy, q charge, $\vec{E}$ the electric field between capacitor plates, and $\Delta\vec{s}$ the separation between capacitor plates. All of this implies that $[q|\vec{E}|] = [E]^2$ and $[qV] = [E]$. Since $|\vec{E}| = \frac{dV}{ds}$, implying, again, that $\frac{[V]}{[\vec{E}]} = [E]^{-1}$, it seems natural to assign $[q]$ to be dimensionless.

1. **What is the natural dimension of force? of acceleration?**

2. **What is the natural dimension of cross-section?**

[1] High-energy projectiles probe small distance and short time scales.

3. **What is the natural dimension of angular momentum?**

4. **What is the natural dimension of magnetic field density, B? [Recall the Lorentz force equation.]**

Theoretical nuclear and particle physicists tend to employ only natural dimensions and units in their calculations. Phenomenologists and especially experimentalists need to make contact with the everyday world and so often employ "hybrid" units, in which most quantities that in natural dimensions have dimension of energy (e.g., mass and momentum) retain an energy dimension, but those that have dimensions not equal to energy to the first power (e.g., time and length) have everyday dimensions. Hybrid systems retain the dimensionless value of one for $\hbar$ and c in expressions that might involve these in everyday dimensions, but hybrid systems employ mixed (hybrid) units of these parameters to translate other quantities from natural to everyday units.

In everyday dimension systems, $\hbar$ has dimensions energy $\times$ time, $[E] \times [T]$, and c has dimensions length / time, $[L]/[T]$. The product $\hbar c$ then has dimensions energy $\times$ length, $[E] \times [L]$.

The natural unit of energy is the electronvolt: 1 eV $= 1.6 \times 10^{-19}$ J (recall that 1 J $=$ 1 kg$\cdot$m^2/s^2). In hybrid units $\hbar \approx 6.6 \times 10^{-16}$ eV$\cdot$s, $c \approx 3 \times 10^8$ m/s, and $\hbar c \approx 2 \times 10^{-7}$ eV$\cdot$m). With these, natural dimensions and units can be converted into everyday dimensions and units, and vice versa.

In everyday units, the (charge) radius of a proton is approximately $r_p \approx 0.8 \times 10^{-15}$ m (or about 1 fm–pronounced "fermi"). Division by $\hbar c$ converts a length in meters into a length in inverse-electon-volts. So, the proton's charge radius in natural units is

$$\begin{aligned}\frac{r_p}{\hbar c} &\approx \frac{0.8 \times 10^{-15}\ \mathrm{m}}{2 \times 10^{-7}\ \mathrm{eV} \cdot \mathrm{m}} \\ &\approx 4 \times 10^{-9}\ \mathrm{eV}^{-1} \\ &= 4\ \mathrm{GeV}^{-1}\end{aligned}$$

In order for a probe to "see" an object, the object must scatter the probe. If the probe is a wave, as are all quantum objects, the wave's wavelength must be roughly the size of the object's diameter (or smaller, of course). So, another way to think about the value of the proton's charge radius is to consider the wavelength of a wave that will be scattered by a proton:

$$\frac{\lambda}{2} \approx r_p \Rightarrow \lambda \approx 2r_p$$

that is, only wave with $\lambda \leq 2r_p$ will "see" the proton. In other words, you measure the diameter of a proton (or other quantum object), by sending ever shorter wavelength probes at it and detecting when the probes begin to scattter. For a proton, this happens at a wavelength of about 2 fm, that is, a probe with energy of greater than 100 MeV $=$ 0.1 GeV (recall that $E \propto \frac{1}{\lambda}$). This energy scale is not far from the proton's mass:

$$
\begin{aligned}
m_p &\approx 1.7 \times 10^{-27} \text{ kg} \Rightarrow \\
m_p c^2 &\approx 1.5 \times 10^{-10} \text{ J} \\
&\approx \frac{1.5 \times 10^{-10} \text{ J}}{1.6 \times 10^{-19} \text{ J/eV}} \\
&\approx 0.940 \times 10^9 \text{ eV} = 940 \text{ MeV}
\end{aligned}
$$

CERN's Large Hadron Collider (LHC) operates at a center-of-mass energy of 13 TeV $= 13 \times 10^{12}$ eV. If, as a rough estimate, each of a proton's three quarks carries about 20% of the proton's energy (the gluons carry most of the rest), then, in principle, new particles of mass $\sim$3 TeV could be produced, if they existed, in proton-proton collisions, and structures of a linear size of about 3×10^{-11} eV^{-1} could be investigated. That is, masses of

$$3 \times 10^{12} \text{ eV} \times 1.6 \times 10^{-19} \text{ J/eV}/9 \times 10^{16} \text{ m}^2/\text{s}^2 \approx 5 \times 10^{-24} \text{ kg}$$

or about $5000 \times m_p$ could be created, and distances of

$$3 \times 10^{-11} \text{ eV}^{-1} \times 2 \times 10^{-7} \text{ eV} \cdot \text{m} \approx 6 \times 10^{-18} \text{ m}$$

or about $r_p/1000$ could be explored.

5. **Show how to convert momentum $\vec{p}$ in everyday dimensions and units into momentum $\vec{p}$ in natural dimensions and units?**

6. **In natural dimensions, an object's kinetic energy is $K = E - m$. Convert this equation into everyday dimensions.**

7. **The everyday dimensions of the universal gravitational constant, G, are**

$$\frac{[L]^3}{[M][T]^2}$$

What is(are) its natural dimension(s)? Its value in Standard International (SI) units is 6.674×10^{-11} m^3/(kg $\cdot$ s^2). What is its value in natural units?

8. **In SI units, Coulomb's Law is**

$$F = \frac{1}{4\pi\varepsilon_0} \frac{Q_1 Q_2}{r^2}$$

   (a) **Argue that the ratio $\frac{Q_1 Q_2}{\varepsilon_0}$ is dimensionless in natural dimensions.**

(b) Recall that $c^2 = \frac{1}{\varepsilon_0 \mu_0}$ and that $c \equiv 1$ and dimensionless in natural dimensions. Argue that setting $\varepsilon_0 = \mu_0 = 1$ and dimensionless implies that charge is dimensionless in natural dimensions, as suggested previously.

(c) Recall that the fine structure constant,

$$\alpha \equiv \frac{e^2}{4\pi\varepsilon_0 \hbar c} = k_E \frac{e^2}{\hbar c} = \frac{1}{137}$$

in all systems of units. k_E is the Coulomb constant. Calculate the charge of the proton, e, in natural units.

(d) Calculate $k_E e^2$ in mixed units of MeV·fm.

9. Consider two nearly touching protons. Compare the strengths of their gravitational and Coulomb interactions. Such a comparison is especially straightforward in natural units.