Week 2: Collisions and Scattering (Graduate)

Nearly everything we know about subatomic objects and their interactions derives from three indirect techniques:

1. measuring the properties of joined or bound entities (spectroscopy)
2. observing what happens when one entity scatters off another
3. examining natural, spontaneous transformations of entities

Spectroscopy primarily investigates energy spectra, while the examination of transformations provides information about lifetimes and the interactions involved in the transformation. We focus here on scattering, which provides information about internal structure. The principle measurables in scattering experiments are cross-sections, both partial and total. A cross-section is the (effective) area for a projectile to interact with a target, a measure of the probability that some process will occur between a projectile and a target. It can be thought of as the transverse size of the target.

In all collisions (of which scattering is an example) within isolated systems, momentum (both linear and angular) and total energy are conserved. In elastic collisions, kinetic energy (in the system) is also conserved.

1. **Consider an elastic collision between two particles. Show that the collision will not change magnitude of either particle's velocity when these are measured in the center-of-mass frame.**

2. **Two particles–one with mass m_A moving at $|\vec{v}_{Ai}|$, the other with mass m_B moving in the opposite direction (to m_A) at $|\vec{v}_{Bi}|$–collide elastically head-on. After the elastic collision, m_A moves at $|\vec{v}_{Af}|$, and m_B moves at $|\vec{v}_{Bf}|$.**

   (a) **Find expressions for $|\vec{v}_{Af}|$ and $|\vec{v}_{Bf}|$ in terms of m_A, m_B, $|\vec{v}_{Ai}|$ and $|\vec{v}_{Bi}|$.**

   (b) **Consider the case of m_B initially at rest. Under what conditions will m_A**

      i. **bounce back;**

      ii. **continue in its initial direction?**

   (c) **Again, when $\vec{v}_{Bi} = 0$, what happens to $|\vec{v}_{Af}|$ and $|\vec{v}_{Bf}|$ when**

      i. **$m_A \gg m_B$;**

      ii. **$m_A \ll m_B$;**

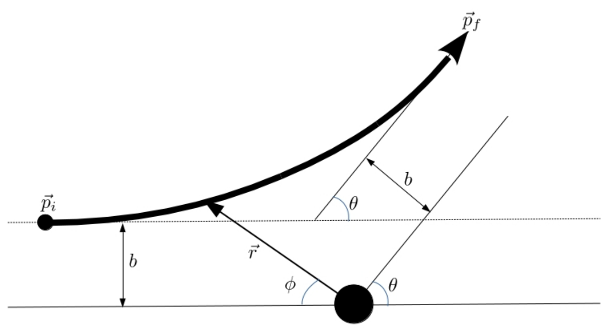
      iii. **$m_A = m_B$?**

      iv. **What happens to kinetic energy under these circumstances?**

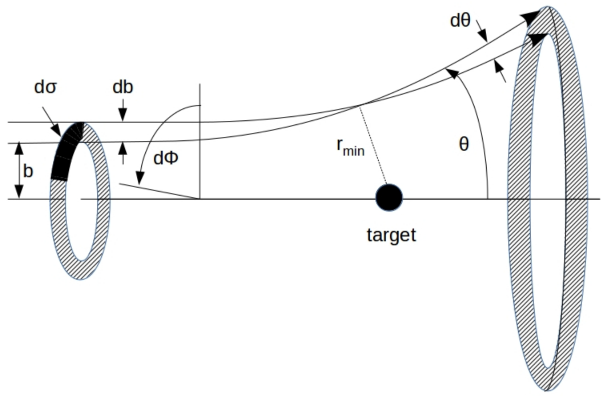
   [Note that m_A always loses kinetic energy in collisions with a stationary object; faster objects always lose kinetic energy in collisions with slower objects.]

In subatomic scattering, no contact is made between particles. Rather, motions change continuously during the interaction, without loss of (total) energy. The motions are conic in geometry, rather than linear. The meanings of elastic and inelastic are then modified, as well. As the result of an inelastic collision, the target breaks apart or occupies a different energy state (becomes excited). A totally inelastic collision is one in which the target completely absorbs the projectile, and then breaks apart or becomes excited. An elastic collision is one in which the projectile emerges from the interaction with the same kinetic it had when it entered.

Subatomic physics scattering experiments employ projectiles with velocities, $v/c \gg 0$. Subatomic targets have dimensions of around 10^{-15} m (1 fm). This implies that an interaction lasts on the order of only 10^{-22} s, implying that, at least in the case of a two-body elastic scatter, the projectile is free both before and after the interaction.



When a subatomic projectile interacts with a subatomic target, the trajectories of both are altered. This is what is meant by a scattering interaction. The projectile's final path differs from its initial path by the polar, or scattering, angle θ. θ is a function, $\theta(b)$, of the impact parameter, b, which is the distance by which the projectile would have missed the center of the target if its trajectory hadn't been deviated. The form of $\theta(b)$ depends on the interaction. Generally, with a symmetric beam, a symmetric target, and a symmetric interaction potential, scattering exhibits no dependence on the azimuthal angle, ϕ.



A projectile approaching a target with an impact parameter between b and $b+db$ emerges with a scattering angle between θ and $\theta+d\theta$. More generally, due to the lack of ϕ-dependence, a projectile passing through the infinitesimal area $d\sigma$ scatters into the corresponding solid angle $d\Omega$. The bigger $d\sigma$, the bigger $d\Omega$:

$$d\sigma = D(\theta)d\Omega$$

The proportionality constant, $D(\theta)$ is called the differential cross section (though it's not a differential).

Because areas and solid angles are positive,

$$\begin{aligned} d\sigma &= |b\ db\ d\phi| \\ d\Omega &= |\sin\theta\ d\theta\ d\phi| \\ \Rightarrow D(\theta) &= \frac{d\sigma}{d\Omega} \\ &= \left|\frac{b}{\sin\theta}\left(\frac{db}{d\theta}\right)\right| \end{aligned}$$

When the projectiles are collimated in a beam aimed at a thin "sheet" of targets, few interactions occur per pass. The number of interactions, N_i, is then proportional to:

- the number of projectiles (in the beam), N_b;
- the number per unit volume (numerical density) of targets, n_t; and
- the thickness of the "sheet", z,

$$N_i = \sigma N_b n_t z.$$

The product $N_b n_t z$ is sometimes referred to as the luminosity, $\mathcal{L}$, so

$$N_i = \sigma\mathcal{L},$$

and the proportionality constant σ is known as the total cross-section. The cross-section gives the probability of an interaction given the characteristics of the beam and target. Its everyday dimension is area, or L^2, and its everyday unit is barn (b): 1 b $= 10^{-28}$ m$^2 \Rightarrow \sim 2.6 \times 10^{-3}$ MeV^{-2} in natural units.

3. If a thin target of thickness z has n_t constituents per unit volume, each of which has a cross-sectional area of σ, what is R, the ratio of the area covered by the constituents to the total cross-sectional area of the target?

4. Recall that the number of scattering interactions in a thin target,

$$\boldsymbol{N_i = \sigma N_b n_t z},$$

where N_b is the number of projectiles, n_t is the number of target constituents per unit volume, z is the target thickness, and σ is the cross-section (the effective scattering area of the target constituents). The number N_f of particles that make it through the target unscattered then is

$$\mathbf{N_f = N_b - N_i}.$$

In a direct total cross section measurement, the number in and the number out are counted to determine the number scattered,

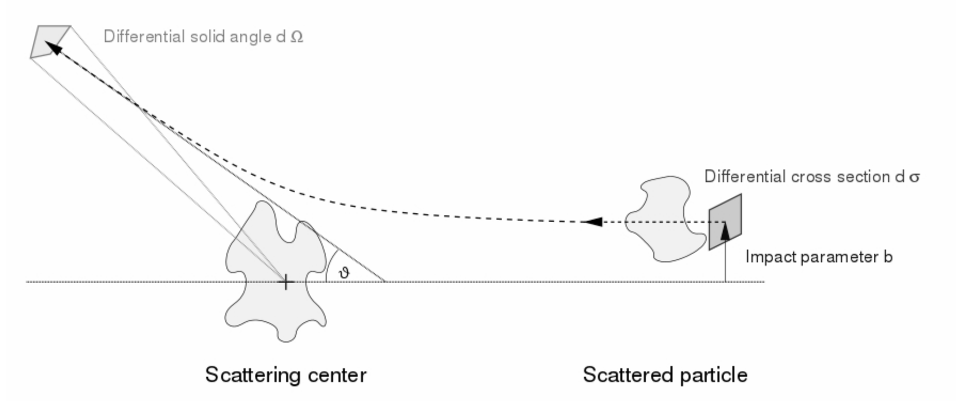
$$\boldsymbol{N_i = N_b - N_f}.$$

For scattering through a not-so-thin target, it is necessary to integrate over a differential thickness, dz. With each differential thickness traversed, the number of unscattered particles, N, decreases, and so N_f at one thickness becomes N_b at the next, and so forth. For such an experiment, then, with a not-so-thin target, the differential number of particles scattered at any given differential thickness will be $dN_i = -dN = \sigma N n_t dz$. By integrating dN from an initial N_b to a final N_f over the entire thickness from 0 to Z, find an expression for the number of particles that go unscattered, N_f. In terms of this result and the total number of projectiles, $\mathrm{N_b}$, find an expression for the total number scattered, N_i.

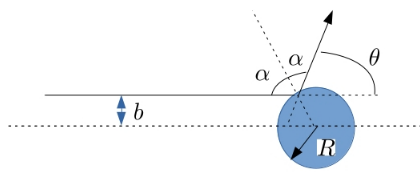
In practice, the term "total cross-section" is somewhat ambiguous. It could mean the sum of cross sections for different types of interactions between beam and target, such as elastic and inelastic. But it could also mean the integral over differential cross-sections, $\frac{d\sigma}{d\Omega}$, at all possible scattering angles:

$$\sigma = \int \frac{d\sigma}{d\Omega} d\Omega$$

where $d\Omega = |\sin\theta\ d\theta\ d\phi|$ is, again, the infinitesimal element of the solid angle. This equals $\frac{dA}{r^2}$, a differential area a distance r from the target, where a detector would be set up.



5. **Consider scattering from a hard sphere of radius R.**



(a) **Show that**

$$b = R\cos\left(\frac{\theta}{2}\right)$$

(b) **What is the derivative of b with respect to θ,**

$$\frac{db}{d\theta}?$$

(c) **Calculate**

$$\frac{d\sigma}{d\Omega}$$

(d) **Finally, calculate**

$$\sigma = \int d\sigma = \int \frac{d\sigma}{d\Omega} d\Omega$$

Does the result make sense?

6. **Non-relativistic Rutherford scattering.**

Central, $1/r$ interactions–those of the form $\frac{K}{r}$–are elastic and conservative. That is, the projectile's mechanical energy and angular momentum are unchanged.

The Coulomb interaction potential can be written

$$U(r) = k_E \frac{Q_p Q_t}{r}$$

where k_E is the Coulomb constant and Q_p (Q_t) is the charge of the projectile (target).

(a) Show that, for the rare case of a head-on collision between a non-relativistic α-particle (${}^4_2\mathrm{He}^{++}$), where $Q_p = 2e$ and e is the charge of the proton, and a nucleus with Z protons, the distance of closest approach is:

$$r_{\min} = \frac{2Zk_E e^2}{K_0}$$

where K_0 is the α-particle's initial kinetic energy.

(b) The most energetic α-particles Ernest Rutherford and his colleagues had available for their scattering experiments were 7.7 MeV. Calculate $r_{\min}$ for

i. gold and
ii. silver targets.

A standard problem in classical mechanics texts asks the student to show that the impact parameter for such an interaction takes the form:

$$b = \left| \frac{k}{2K_0} \right| \cot\left(\frac{\theta}{2}\right)$$

where k is the numerator of the central force equation and K_0 is, again the projectile's initial kinetic energy.

For a Coulomb interaction, this becomes

$$b = \left| k_E \frac{Q_p Q_t}{2K_0} \right| \cot\left(\frac{\theta}{2}\right) = \left| \frac{Z k_E e^2}{K_0} \right| \cot\left(\frac{\theta}{2}\right)$$

(c) Derive the expression for

$$\frac{d\sigma}{d\Omega}$$

(d) Calculate and interpret the total cross section.

7. If 7.7 MeV kinetic energy α-particles scatter at 90° relative to its original direction from a Uranium 238 nuclei (${}^{238}_{92}\mathrm{U}$), initially at rest, find:

(a) The scattering angles of the α-particle and Uranium nucleus in the center-of-mass frame.

(b) The recoil scattering angle of the Uranium nucleus in the lab frame.

(c) The kinetic energies of the scattered α-particle and Uranium nucleus (in MeV) in the lab frame.

(d) The impact parameter, b.

(e) The differential scattering cross-section at 90°.

8. How much kinetic energy must an α-particle have to just penetrate a silver ($^{107}_{47}Ag$) nucleus?