Week 6: Nuclei (Undergraduate)

[*A table of masses of the elements will be useful for many of the problems in this exercise set.*]

An atomic nucleus is comprised of one or more positively-charged protons and, except in the case of hydrogen, at least one neutron, which, as the name suggests, is neutral. Protons and neutrons, collectively called nucleons, have roughly the same mass (938.3 MeV and 939.6 MeV), approximately 1900 times greater than that of electrons (0.511 MeV). The magnitude of the proton's charge is exactly equal to that of the electron, but of opposite sign.

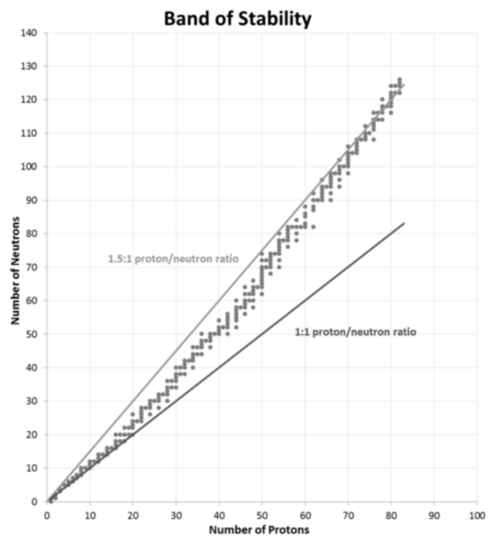
The number of protons in a nucleus is referred to as the atomic number and designated Z, the number of neutrons is referred to as the neutron number and designated N, and the total number of nucleons is referred to as the mass number and is designated $A = Z + N$. Often, nuclear masses are given in terms of u, the unified mass unit, where $1\ u \equiv \frac{1}{12}$(mass of the ${}^{12}_{6}$C atom).

1. **What is the value of 1 u in natural units? [Recall that 1 mole of ${}^{12}_{6}$C is 12 grams, and Avogadro's number is $N_A \equiv 6.02214076 \times 10^{23}\ \mathrm{mol}^{-1}$.]**

Each nucleus with a certain Z and a certain N is know as a species or nuclide, of which around 340 occur naturally. More than 3000 additional nuclides have been produced artificially. Nuclides are represented symbolically as ${}^{A}_{Z}$X, where X is the chemical symbol for the element containing Z protons. Isotopes of an element all have Z protons but different $N = A - Z$ neutrons and, consequently, different A. Isotones have the same N but different Z, and isobars have the same A, but different Z and N.

2. **How many protons (Z) and neutrons (N) do each of the following nuclei have:**
   - **(a)** ${}^{9}_{4}$**Be**
   - **(b)** ${}^{12}_{6}$**C**
   - **(c)** ${}^{23}_{12}$**Mg**
   - **(d)** ${}^{40}_{20}$**Ca**
   - **(e)** ${}^{43}_{20}$**Ca**
   - **(f)** ${}^{54}_{26}$**Fe**
   - **(g)** ${}^{56}_{26}$**Fe**
   - **(h)** ${}^{57}_{26}$**Fe**
   - **(i)** ${}^{197}_{79}$**Au**

The neutron number has large effects on nuclear properties, but its effect on chemical properties is negligible for most elements. Ordinarily, atoms are neutral, having the same number of electrons as protons. An element's chemical behavior is primarily determined by its electronic structure. Electronic structure depends on the number of electrons–and therefore the number of protons–along with the Pauli exclusion principle. Different isotopes of a given element have the same number of protons, and therefore, electrons, and so share a similar electronic structure. Consequently, the chemistry of the isotopes of an element are nearly identical. This has biological consequences if one or more isotopes of a biologically significant element are radioactive.



The numbers of protons and neutrons in light nuclei are roughly equal. That is, in light nuclei, $N \approx Z$. As the number of nucleons increase, however, stability seems to require an ever greater fraction of neutron. That is, in heavier nuclei, $N > Z$. Additional neutrons keep protons farther from one another, reducing their mutual Coulomb repulsion. Because of this repulsion, neutrons are more densely packed than protons in the nucleus, so, for a given volume, there can be more of them.

Assuming the overall nuclear density of a given nucleus is approximately constant, then nuclear volume will be directly proportional to A, the number of nucleons. The (charge) radius of a nucleus can be determined by firing positively charged projectiles, such as alpha particles (nuclei of helium atoms), at a sample of the element of interest and noting the scattering pattern. If the scattering results from the Coulomb interaction, then the differential scattering cross-section (the number of scatters into a given solid angle) follows a $1/\sin^4(\theta/2)$ distribution (that is, Rutherford scattering). The maximum projectile energy which follows this pattern yields the charge radius:[1]

[1] Note that the projectile velocities are not relativistic, so Newtonian mechanics is sufficient:

$$\frac{1}{2}mv^2 = \frac{1}{4\pi\epsilon_0}\frac{Q_1 Q_2}{r}$$

$$r = r_{\min} = \frac{2Q_1Q_2}{4\pi\epsilon_0 mv^2}$$

By increasing their energy, projectiles can reach the interior of a nucleus, at which point the scattering looks different. The maximum energy before this happens yields the "charge radius" of the nucleus. Experiments suggest that the charge radius depends on the mass number, A, and given approximately by

$$r = r_0 A^{1/3}$$

where $r_0 = 1.2 \times 10^{-15}$ m $= 1.2$ fm, as determined empirically. The mass radius follows the same form, but, in this case, $r_0 = 1.4$ fm.

Note that the volume of the nucleus is proportional to r^3 and so proportional to the mass number, A, consistent with fair uniformity of density among nuclei. Note, too, that the radius of atoms is $\sim 10^{-10}$ m $= 1$ Å. That is, the linear dimension of a nucleus is 10^5 times smaller than that of an atom.

3. **How many times larger is the radius of a gold ($^{197}_{79}$Au) nucleus than that of a helium-4 ($^{4}_{2}$He) nucleus?**

4. **Estimate (the typical) nuclear density in kg/m^3. For reference, the density of a chunk of lead $^{209}_{82}$Pb is about 10^4 kg/m^3.**

In order for nuclei to exist as tiny, dense cores of atoms, some interaction must overcome the electrostatic repulsion between protons. That interaction has come to be referred to as the strong (nuclear) interaction (as we know, gravitational attraction between protons is many orders of magnitude weaker than their electrostatic repulsion). Experiment shows that at separations of around 1 fm, the strong interaction attracts neutrons to neutrons and neutrons to protons as strongly as it attracts protons to protons, yet it has no effect on electrons or photons. Though strong, its range is limited to about 2 fm.[2] At separations of much less than 1 fm, the interaction becomes repulsive.

Once bound together, work is required to separate the nucleons of stable atoms. The amount of work required is referred to as the binding energy, E_b. A result of special relativity is that the mass of a system with a positive binding energy is smaller–the mass deficit, Δm–than the sum of the masses comprising the system.

The velocity can be determined with perpendicular electric and magnetic fields:

$$qE = qvB \Rightarrow v = E/B$$

[2]This is due to the fact that the strong interaction is directly between quarks of which nucleons are composed. The nucleon-nucleon interaction is effectively similar to a van der Waals interaction that binds certain molecules. From a distance, the nucleon appears neutral with respect to the strong interaction between quarks, just as the atom appears neutral from a distance despite the electric field between nucleus and electrons.

$$\begin{aligned} E_b &\equiv E_{\text{parts}} - E_{\text{sys}} \\ \Delta m &= m_{\text{parts}} - m_{\text{sys}} = E_b \quad \text{(in natural dimensions)} \end{aligned}$$

For the specific case of a nucleus (ignoring the much smaller electron binding energies):

$$\Delta m = Z m_H + N m_n - m_{\text{atom}}$$

where Z is the proton number, N is the neutron number, $m_H = 1.007825u = 938.7833$ MeV is the mass of a hydrogen atom (including the electron mass), $m_n = 1.008665u = 939.5654$ MeV is the mass of a neutron, and m_{atom} is the mass of the atom. (The proton's mass is $m_p = 1.007276u = 938.2721$ MeV.)

The amount of energy required to remove a nucleon from a nucleus (equivalent to ionizing an atom) is related to the binding energy:

$$m_{\text{original nucleus}} + E_b = m_{\text{final nucleus}} + m_{\text{nucleon}}$$

5. **Compute the mass deficit in unified mass units (u) and the binding energy in MeV of the following atoms:**
   (a) **The mass of $^{12}_{6}$C is exactly 12 u by definition.**
   (b) **The mass of $^{23}_{11}$Na is 22.989770 u.**
   (c) **The mass of $^{40}_{20}$Ca is 39.962591 u.**

6. **The helium-4 ($^{4}_{2}$He) atom is composed of four nucleons.**
   (a) **Calculate helium-4's binding energy and binding energy per nucleon**
   (b) **In the helium-4 nucleus, each nucleon is in contact with the others. There are therefore six two-nucleon interactions, and the binding energy of the atom is the sum over pairs of each two-nucleon interaction. Assuming each interaction is equivalent, compare the strong interaction potential (because it's attractive, it will be negative–by definition, the potential is zero at infinity) between two nucleons with the Coulomb potential between two protons whose centers are separated by 1 fm.**

7. **How much energy is required to remove from a $^{40}_{20}$Ca nucleus the least tightly bound (i.e., the first to be removed)**
   (a) **neutron**
   (b) **proton.**
   (c) **Explain the difference**

A deuteron (called deuterium with an electron) is the simplest compound nucleus. It comprises one proton and one neutron. Its charge is the same as the proton's, while its mass is 1875.6129 Mev = 2.014 u. It has a non-zero electric quandrupole moment, which means it's not spherical, and it does not possess a definite angular moment. Already in the deuteron, with just two nucleons, the complexity of nuclear structure is manifest.

8. What are the deuteron's binding energy (in MeV) and mass deficit in (u)?