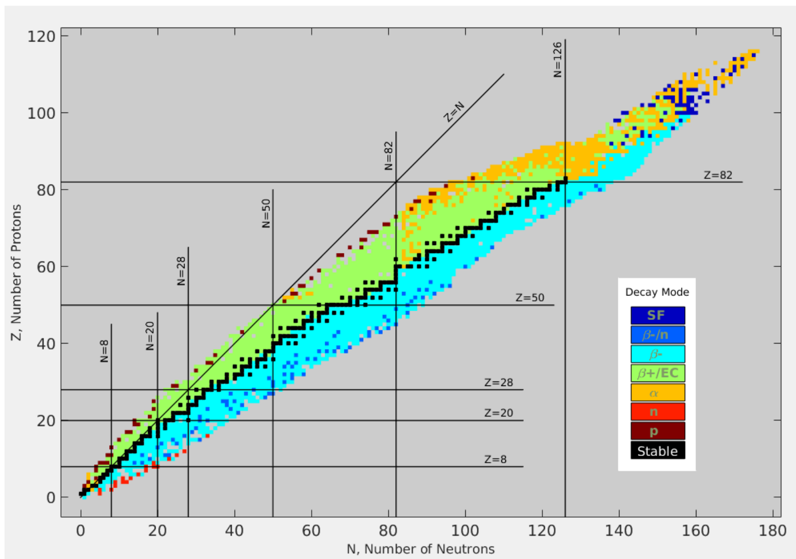
Week 7: Nuclear Stability and Instability (Graduate)

Although the strong interaction overcomes electrostatic repulsion, it only does so under certain conditions and, generally, for only a certain amount of time. Only a limited set of nucleon configurations results in stable nuclei; for most other possible configurations, the nucleus transforms after some time into a more stable configuration (shedding pieces) or never forms in the first place.



The figure raises questions requiring explanation [note that, in this figure, the axes are reversed with respect to the equivalent graph last week]:

1. $N = Z$ for small A, but $N > Z$ for large A?
2. Only a small fraction of nuclei are stable.
3. No stable nuclei with $Z > 83$?
4. Unstable nuclei transform in a variety of ways after a variety of time intervals?
5. Transformation mechanisms vary, but systematically?

Addressing here the fourth and fifth questions, three types of transformation dominate: alpha (involving $^4_2$He nuclei), beta (involving electrons or positrons), and gamma (involving high-energy photons).

Experiment indicates that the rate at which a sample of some unstable substance transforms decreases exponentially with time. This rate is also referred to as the activity, A (not to be confused with atomic mass).

$$A(t) = A_0 e^{-t/\tau}$$

where A_0 is the activity at $t = 0$, $\tau \equiv t_{1/2}/\ln 2$ is called the (mean, or average) lifetime or (exponential) time constant, and $t_{1/2}$ is the amount of time it takes a sample's activity to decrease by 50% ($A_0 \to A_0/2$), also known as the half-life. Note that τ and therefore $t_{1/2}$ characterize the unstable substance; the half life and lifetime are unique to the substance.

1. **The half-life of $^{28}_{13}Al$ is 2.24 minutes. What fraction of the substance remains after 1 hr?**

2. **Exponential decay implies that no matter how much time has passed since the activity began, after another half-life, the activity will still decrease by a factor of 2.**

   (a) **Demonstrate that $A(t + t_{1/2}) = A(t)/2$ is independent of t.**

   (b) **Argue that the choice of $t = 0$ is therefore arbitrary.**

3. **The number, N, of constituents making up a substance undergoing a transformation becomes smaller at the rate of the transformation (activity), $A = -dN/dt$. That is, the activity during the interval dt equals the number of constituents that transform (the negative sign indicates that N is decreasing, because $A \geq 0$).**

   (a) **Prove that $A = N/\tau$.**

   (b) **Argue that if, instead of decreasing exponentially, A remained constant, then the sample would completely transform in time τ (hence, the name lifetime).**

4. **When a quantum system transforms from a higher energy state to a lower energy state, the transformation probability during the infinitesimal interval dt remains constant. That is, the transformation probability is independent of time during the infinitesimal interval dt. The transformation probability is therefore proportional to dt (the bigger dt, the greater the transformation probability). Let the proportionality constant, which is characteristic of the substance, be λ, known as the decay constant (not to be confused with wavelength). The transformation probability for a given substance is therefore λdt.**

   If the number of remaining constituents of a transforming substance is $N(t)$ at time t,

   (a) **Argue that $dN/N = -\lambda dt$. Explain the $-$ sign.**

   (b) **By integrating both sides of this equation, show that $\ln(N/N_0) = -\lambda t$. Explain N_0.**

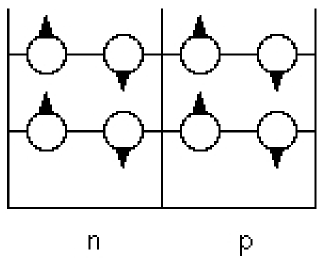
   (c) **Taking the exponential of both sides of the equation leads to $N(t) = N_0 e^{-\lambda t}$. In words: equal transformation probability throughout an infinitesimal interval implies an exponential** decrease in the number untransformed constituents of an unstable sample. Recall that A is the transformation rate. Argue that $A = \lambda N$.**

(d) **Argue that $N(t) = N_0 e^{-\lambda t}$ implies $A(t) = A_0 e^{-\lambda t}$.**

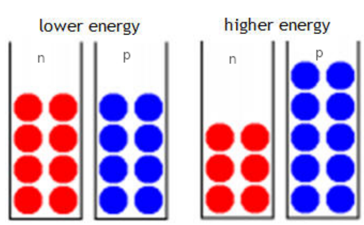
(e) **How is λ related to τ and $t_{1/2}$?**

(f) **Show that the average lifetime of an unstable nucleus is inversely related to λ:**

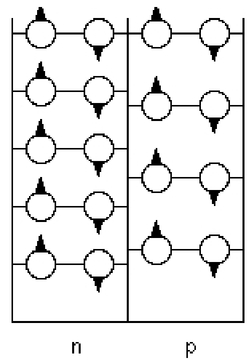
$$t_{\text{ave}} = \frac{1}{\lambda}$$



With regard to the first and second questions (regarding the stability of nuclei and the ratio of protons to neutrons in a nucleus), consider a nucleus as a square well populated by nucleons. The Pauli exclusion principle suggests that no more than four nucleons can occupy any energy level: two protons of opposite spin and two neutrons of opposite spin. For a nucleus with few nucleons, the lowest energy levels of protons and of neutrons are approximately equal, and the levels are roughly equally spaced, so, if $Z > N$ and if there are fewer than two neutrons in the highest neutron level below the highest proton level, the nucleus is in a higher energy state than it would be if the numbers of protons and neutrons were more equal. The nucleus is then unstable. It will eventually transform into a nucleus composed of more equal numbers of protons and neutrons.



On the other hand, the electrostatic repulsion in a many-proton nucleus increases the protons' energy levels (both of the ground state and of the spacing between levels), so that more neutrons than protons are necessary to balance energy. Without the extra neutrons, the nucleus will be unstable



Of course, a nucleus with an excess of neutrons above the highest proton energy level is also unstable, so, at some point many neutrons can become too many neutrons.

To summarize: a nucleus will be unstable and will fall apart or transform if the ratio Z/N is not optimal for the number of nucleons present, or if there are too many protons, or if one or more nucleons occupies an energy level above the lowest available energy level. Under any of these conditions, the nucleus, if it forms at all, will eventually undergo alpha, beta, or gamma transformations.

5. **$^{13}_{8}$O transforms to $^{13}_{7}$N (emitting a positron and a neutrino) with a half-life of 8.9 ms. Explain what is going on and why.**

6. **$^{12}_{7}$N is unstable. Explain why and describe what will happen to it.**

7. **Although $Z = N$, $^{18}_{9}$F is unstable. Explain why and describe what will happen to it.**

8. **(a) Although $Z = N$, $^{44}_{22}$Ti is unstable. Explain why and describe what will happen to it.**

   (b) The nucleus produced by the $^{44}_{22}$Ti transformation is also unstable. Explain why and describe what will happen to it.

9. **$^{41}_{20}$Ca is unstable. Predict what happens to it.**