Week 8: Nuclear Transformations and Radioactivity (Graduate)

[*A table of masses of the elements will be useful for many of the problems in this exercise set.*]

Unstable nuclei transform into more stable nuclei, emitting particles (maybe including photons), as long as energy, (linear and angular) momentum, total mass number, and electric charge are conserved. Nuclear transformations conserve the number of nucleons, a corollary of a more fundamental conservation law, that of baryon number, which will be discussed in the study of particle physics.

The transformations are traditionally referred to as decays, in particular, radioactive decays. They follow the statistical radioactive decay law introduced last week.

A nucleus that doesn't fall apart, but is nonetheless unstable, usually transforms via alpha, beta, or gamma transformations, in which the unstable nucleus (referred to as the parent) transforms into another sort of nucleus (referred to as the daughter) with the emission of an alpha particle (helium-4 nucleus, ${}^4_2\mathrm{He}$), a beta particle (an electron or a positron), or a gamma-ray (a high-energy photon).

Nucleons in excited energy states (after a transformation or other interaction) may, as do atoms, relax to lower energy states by emitting photons with energies equal to the difference between energy states:

$$E_\gamma = h\nu = E_i - E_f.$$

Such photons from nuclear energy-state transitions are approximately a million times more energetic than the photons emitted in atomic (electron energy-level) transitions. With energies in the MeV range (rather than the eV range), these photons are referred to as gamma rays, and the transformation is sometimes called a gamma decay. Gamma rays are one form of radiation.

As gamma rays are chargeless and massless, a gamma transition leaves Z and A (and therefore N) unchanged. An excited state is typically designated by an asterisk, $[{}^A_Z\mathrm{X}]^*$, so a gamma transition would be written symbolically as

$$[{}^A_Z\mathrm{X}]^* \rightarrow {}^A_Z\mathrm{X} + \gamma$$

The half-life of a gamma transition is normally around 10^{-14} s, but can be much longer. Excited nuclei with half-lives in excess of 10^{-9} s are considered metastable and called isomers (they have the same Z, N, and A before and after the transformation). Their excited states are called isomeric states. Isomeric states can last from nanoseconds to centuries, depending on the type of nucleus.

While the strong nuclear interaction overcomes electrostatic repulsion between protons, its range is short, on the order of 1 fm (10^{-15} m). The electrostatic interaction, by contrast, falls off as $1/r$, and so extends to infinity. Thus, the strength of the strong interaction in holding the nucleus together is roughly proportional to the number of nucleons, while the electrostatic interaction pushing protons apart is roughly proportion to the atomic number squared. At around $A = 210$, the two interactions are approximately equal. The transformation of such large nuclei, typically via the alpha transformation, reduces A by 4 and Z by 2, and tends to result in more stable nuclei.

Why preferentially emit a helium nucleus? Wouldn't a proton, neutron, or other light nuclei be just as, if not more, likely? One reason is that helium-4's nuclear binding energy, at 7 MeV per nucleon, is significantly greater than that of other light nuclei, and so is more stable than they are. Emitting an alpha particle therefore releases a good deal more energy than other possible alternatives. In fact, energy may need to be added in order to emit some of these alternatives.

Denoting, then, the parent as P and the daughter as D, the transformation can be written symbolically as:

$$^{A}_{Z}P \rightarrow \, ^{A-4}_{Z-2}D + \, ^{4}_{2}\mathrm{He}$$

or

$$^{A}_{Z}P \rightarrow \, ^{A-4}_{Z-2}D + \alpha$$

In the rest frame of the parent, conservation of energy requires

$$M_P = M_D + m_\alpha + K_D + K_\alpha$$

where M_P, M_D, and m_α are the parent, daughter, and α-particle masses, and K_D and K_α are the daughter and α-particle kinetic energies. Capital M indicates atomic mass (including nucleus and electrons, minus the atomic binding energy, which is quite small for electrons). Kinetic energy cannot be negative, so only if $M_P \geq M_D + m_\alpha$ will an alpha transformation occur.

The sum of the daughter and alpha-particle kinetic energies is referred to as the disintegration energy, Q, the energy released from the difference between parent and daughter and alpha-particle masses.

$$Q = K_D + K_\alpha = M_P - M_D - m_\alpha$$

The alpha transformation requirement that $M_p \geq M_D + m_\alpha$ implies that, when $Q > 0$, the alpha transformation can occur.

1. **Is $^{236}_{94}$Pu stable or will it transform via an alpha transformation? Explain.**

2. **$^{232}_{92}$U is unstable. Explain quantitatively why it transforms to a lower energy state by emitting an alpha particle rather than a proton, a neutron, or a deuteron (the nucleus of deuterium, $^{2}_{1}$H). [Calculate Q.]**

The disintegration energy is fixed for any alpha transition of a particular species of nucleus. It's also frame-independent: all observers measure the same value.

The kinetic energies are small compared to the mass of the parent, so Newtonian mechanics works fine: $K_D = \frac{p_D^2}{2M_D}$ and $K_\alpha = \frac{p_\alpha^2}{2m_\alpha}$ In the parent nucleus's rest frame, the daughter and alpha-particle momenta are equal in magnitude, so

$$\frac{K_D}{K_\alpha} = \frac{m_\alpha}{M_D} \approx \frac{4}{A-4} \tag{1}$$

where here A is the parent's mass number. Then,

$$Q = K_D + K_\alpha \approx \frac{A}{A-4} K_\alpha \tag{2}$$

and therefore

$$K_\alpha \approx \frac{A-4}{A} Q \tag{3}$$

and

$$K_D \approx 4\frac{Q}{A} \tag{4}$$

Thus, the helium nuclei emitted by an alpha transformation are monoenergetic, as are the daughter particles. The larger the parent's mass number, the smaller is K_D and the closer K_α comes to equaling Q.

3. **Derive equations 1, 2, 3, and 4.**

4. **Assuming the $^{232}_{92}$U is at rest when it transforms via an alpha transition, what is the kinetic energy of the alpha particle?**

5. $^{144}_{60}\mathrm{Nd} \rightarrow ^{140}_{58}\mathrm{Ce} + \alpha$. **What are the kinetic energies of the Cerium nucleus and the alpha particle?**

Three processes transform Z but not A: electron emission, positron emission, and inner electron capture. The first of these is negative beta (β^-) decay, in which a neutron transforms into a proton, emitting an electron and an electron anti-neutrino:

$$n \rightarrow p + e^- + \overline{\nu}_e$$

In a nucleus, this becomes:

$$^A_Z P \rightarrow \ _{Z+1}^{\ \ A} D + e^- + \overline{\nu}_e$$

A typical example of negative beta decay is

$$^{12}_5\mathrm{B} \rightarrow \ ^{12}_6\mathrm{C} + e^- + \overline{\nu}_e$$

The second process is positive beta (β^+) decay, in which a proton transforms into a neutron, emitting a positron and an electron neutrino. Note that a free proton cannot undergo this process, since $m_p < m_n$, but the process can occur in a nucleus in the case where the daughter's binding energy exceeds that of the parent, the additional released energy going to the the creation of the neutron, positron, and electron neutrino. In nuclear notation, this is

$$ {}^{A}_{Z}P \rightarrow {}^{\ \ A}_{Z-1}D + e^+ + \nu_e $$

and a typical example transformation is

$$ {}^{12}_{7}\mathrm{N} \rightarrow {}^{12}_{6}\mathrm{C} + e^+ + \nu_e $$

The third process is electron capture, another way a proton is transformed into neutron, this time by a proton absorbing an inner shell electron, after which only a neutrino is emitted. In nuclear notion,

$$ e^- + {}^{A}_{Z}P \rightarrow {}^{\ \ A}_{Z-1}D + \nu_e $$

and a typical example is

$$ e^- + {}^{7}_{4}\mathrm{Be} \rightarrow {}^{7}_{3}\mathrm{Li} + \nu_e $$

The capture is followed shortly afterward by the emission of an X-ray photon as the inner shell is refilled by an outer shell electron. The wavelength of the X-ray is of course characteristic of the daughter atom (not the parent atom).

The neutrino mass is so small even compared to the electron mass that it can be ignored in considering energy conservation, which is then expressed, in the parent's rest-frame:

$$ M_P = M_D + m_e + K_{\text{total}} $$

where we take into account that a particle and its anti-particle have exactly the same mass.

The threshold disintegration energy, Q, for one or another of these processes can be determined in terms of the nuclear masses involved, but only after including electron masses.[1]

For β^- decay:

$$ {}^{A}_{Z}P \rightarrow {}^{\ \ A}_{Z+1}D + e^- + \overline{\nu}_e $$

$$ \begin{aligned} M_P - Zm_e &= M_D - (Z+1)m_e + m_e + Q \\ \Rightarrow Q &= M_P - M_D \end{aligned} $$

That is, negative beta decay can occur if the parent (atomic) mass is greater than the daughter (atomic) mass.

[1] The mass of an electon is $m_e = 9.11 \times 10^{-31}$ kg $= 0.511$ MeV $= 0.00054858$ u.

For β^+ decay:

$$ {}^{A}_{Z}P \rightarrow {}_{Z-1}^{A}D + e^+ + \nu_e $$

$$ \begin{aligned} M_P - Zm_e &= M_D - (Z-1)m_e + m_e + Q \\ \Rightarrow Q &= M_P - M_D - 2m_e \end{aligned} $$

That is, positive beta decay can occur if the parent (atomic) mass is greater than the sum of the daughter (atomic) mass and two electron masses.

For electron capture:

$$ e^- + {}^{A}_{Z}P \rightarrow {}_{Z-1}^{A}D + \nu_e $$

$$ \begin{aligned} M_P - Zm_e + m_e &= M_D - (Z-1)m_e + Q \\ \Rightarrow Q &= M_P - M_D \end{aligned} $$

That is, electron capture can occur if the parent (atomic) mass is greater than the daughter (atomic) mass.

Electron capture is a competing process for nuclear proton-to-neutron transformation if the mass difference between parent and daughter is sufficient for positive beta decay. It is the primary process when that difference, if still positive, is insufficent for positive beta decay.

Note that electrons and positrons do not exist in the nucleus. They are created or absorbed when a nucleus transforms to a lower energy state from a higher energy state.

6. **Describe, qualitatively, the energy spectrum of the neutrino emitted in an electron capture transition.**

7. **An isotope of beryllium at rest transforms into an isotope of lithium via electron capture:**

$$ e^- + {}^{7}_{4}\mathbf{Be} \rightarrow {}^{7}_{3}\mathbf{Li} + \nu_e $$

Determine the energies and momenta of the daughter lithium isotope and electron neutrino. [Find the Q-value from the parent beryllium and daughter lithium's rest masses. As a first approximation, assign it entirely to the nearly massless neutrino. Because the parent beryllium is at rest, the magnitudes of the lithium daughter and neutrino's momenta are equal. Then find the lithium daughter's kinetic energy with the non-relativistic relationship between momentum and kinetic energy. Is the first approximation assignment of the Q-value to the neutrino's energy reasonable?]

8. (a) **The ${}^{6}_{2}$He nucleus is unstable. How does it transform? [There could be more than one way.]**

   (b) **The ${}^{14}_{8}$O nucleus is unstable. How does it transform? [There could be more than one way.]**

9. (a) **The atomic mass of ${}^{60}_{27}$Co is 59.933819 u, and the atomic mass of ${}^{60}_{28}$Ni is 59.930788 u. Which of these nuclei transforms into the other? How?**

   (b) **The atomic mass of ${}^{79}_{35}$Br is 78.918336 u, and the atomic mass of ${}^{79}_{36}$Kr is 78.920084 u. Which of these nuclei transforms into the other? How?**

10. **${}^{55}_{26}$Fe has an atomic mass of 54.938296 u. ${}^{55}_{25}$Mn has an atomic mass of 54.938047 u. ${}^{55}_{26}$Fe transforms into ${}^{55}_{25}$Mn with a half-life of 2.7 y. Does it do so via positive beta decay or electron capture? Explain.**

11. **${}^{81}_{34}$Se has an atomic mass of 80.917992 u. ${}^{81}_{35}$Br has an atomic mass of 80.916291 u. ${}^{81}_{36}$Kr has an atomic mass of 80.916592 u. ${}^{81}_{37}$Rb has an atomic mass of 80.918996 u. One of them is stable. Which is it? How do the others transform?**

The three transformations that change Z but not A, negative beta decay, positive beta decay, and electron capture, are manifestations of the weak interaction, so-called because it is weaker than the strong and electromagnetic interactions, though stronger than the gravitational interaction. Unlike the strong interaction, which occurs between only quarks, not leptons, the weak interaction occurs between both quarks and leptons. Like the strong interaction, but unlike the electromagnetic interaction, the weak interaction occurs between both charged and neutral particles. The range of the gravitational and electromagnetic interactions, being $1/r$ interactions, is infinite. The gravitational interaction is always attractive, while the electromagnetic interaction can be attractive or repulsive depending on the signs of the charges assocated with the interacting particles. The range of the strong interaction is roughly 1 fm. At this distance, the strong interaction is highly attractive. Beyond that, it weakens quickly. It also weakens and even reverses direction at distances less than a proton radius (about 0.8 fm). The range of the weak interaction is less than 0.1 fm and it strengthens at shorter distances, approximately equaling that of the electromagnetic interaction at 1 am (attometer, 10^{-18} m).

The weak interaction frequently involves neutrinos or anti-neutrinos. In fact, for all intents and purposes, the weak interaction is the only one involving neutrinos and anti-neutrinos. Most importantly, however, the weak interaction is almost solely responsible for transforming fundamental particles from one type to another.

12. **The neutrino was first detected in 1956. Prior to that, in 1931, Wolfgang Pauli proposed its existence as a solution to a puzzle:**

in beta decays (both negative and positive), the measured kinetic energy of the emitted charged particle forms a broad spectrum in which nearly all particle energies were lower than that expected of a two-body decay (in a two-body decay of a parent at rest, the momenta of the daughter particles must be equal in magnitude and oppositely directed, so the kinetic energy is uniquely determined).

(a) Explain how Pauli's introduction of a third, neutral, (approximately) massless particle can explain the broad kinetic energy spectra of the electron/positron products of beta decay.

(b) $^{210}_{83}$Bi transforms into $^{210}_{84}$Po. The atomic mass of $^{210}_{83}$Bi is 209.984120 u, and the atomic mass of $^{210}_{84}$Po is 209.982874 u. What is the maximum kinetic energy of the neutrino emitted in the transformation? (Assume the neutrino is massless.)