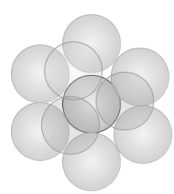
Week 9: Nuclear Models (Graduate)

Experiments reveal quantized nuclear energy levels, demonstrating that nuclei are quantum objects. Nevertheless, no theory of nuclei comparable to that describing hydrogenic atoms has been formulated. A quantum field theory (merging quantum mechanics with special relativity) called Quantum Chromodynamics (QCD) describes the strong interaction between quarks well (at least under certain circumstances), but it has yet to be applied satisfactorily to nuclei, which are complex, many-body objects compared to quarks, and where much of the action is at non-relativisitic speeds. Furthermore, while the characteristic energies of atomic phenomena are on the order of eV, the characteristic energies of nuclear phenomena are a million times greater, on the order of MeV. Thus, while atoms are easily excited and combine readily into molecules and crystals, nuclei excite under only special circumstances. So even quantum mechanics is unnecessary to understand much (but far from all) of what goes on in nuclei. As far as a theory of nuclei is concerned, then, there exists a number of models, some based on QCD, but each describes only a subset of nuclear properties.

One example of such a model, called the liquid drop model, is based on an analogy with an incompressible drop of liquid whose density is constant, and whose size and heat of vaporization (analogous to the binding energy) are proportional to its mass, or, equivalently, the number of its constituents. Its primary virtue is that it accounts for the binding energy of surface nucleons, which is smaller than that of interior nucleons.

The first assumption of the model is that each pair of interior nucleons contributes a characteristic binding energy, ϵ_b. The next assumption is that the nucleus is a tightly packed, perfect sphere (which is a pretty good approximation for large nuclei), so each internal nucleon is in contact with 12 other nucleons–6 in the same plane, 3 above, and 3 below. Each internal nucleon therefore contributes $12\epsilon_b$ to the total binding energy.



A nucleon on the surface has no neighbors above, however, and so is in contact with just 9 other nuclei, assuming, again, perfect sphericity. Surface nucleons therefore contribute $9\epsilon_b$ to the total binding energy.

1. **The semi-empirical mass formula:**

   (a) **If A_{int} is the number of internal nuclons, then the number of surface nuclons is $A_{\text{surf}} = A - A_{\text{int}}$, the total number of nucleons minus the number of internal nucleons. Find an expression of the nuclear binding energy, E_b, in terms of**

A, $\mathrm{A_{surf}}$, and ϵ_b, the binding energy per nucleon pair. (How many nucleons are there and how much does each contribute to the total binding energy?)

(b) Recall that the empirical formula for nuclear radii as a function of mass number is $r = r_0 A^{1/3}$. This suggests that a nucleon is an incompressible ball of approximate radius r_0, and implies that the average radial distance of the center of the nuclear surface layer from the center of the nucleus is $R = r - r_0 = r_0(A^{1/3} - 1)$. The number of surface nucleons A_{surf} can therefore be approximated by dividing the volume of the surface layer [the product of the area of the surface layer (surface area of a sphere) and the thickness of the surface layer (the diameter of a nucleon)] by the volume of one nucleon. Find an approximate expression for A_{surf} in terms of the mass number, A.

(c) Combine the two expressions to find an approximate expression for the binding energy in terms of total number of nucleons, A, and the binding energy per nucleon pair, ϵ_b.

(d) This result considers only binding due to the strong interaction. Because there are protons, binding is reduced by electrostatic repulsion, which is not limited to nearest-neighbor interactions (as strong interaction binding is) and every proton interacts electrostatically with every other proton in the nucleus. That is, there are $\frac{1}{2}Z(Z-1)$ interactions (each of the Z protons interacts with the other $Z-1$ other protons; the $\frac{1}{2}$ ensures that only one interaction per pair is counted). Still assuming only large nuclei, so that $Z(Z-1) \approx Z^2$, and averaging over $1/r$, explain that the electrostatic contribution to the binding energy is

$$E_b \approx -\frac{Z^2 ke^2}{2}\left(\frac{1}{r}\right)_{\mathrm{ave}}$$

Explain the negative sign.

(e) Look back at the figure in the week 5 exercise set. Notice that stable large nuclei tend to have roughly three neutrons for every two protons, thus $N \approx \frac{3}{2}Z$ and, therefore, $A = N + Z \approx \frac{5}{2}Z$ or $Z \approx \frac{2}{5}A$. Moreover, the distance between proton centers ranges from $2r_0$ to $2r_0A^{1/3}$, so the average separation between proton centers is roughly one nuclear radius: $\Rightarrow (1/r)_{\mathrm{ave}} \approx 1/r_0 A^{1/3}$. Using these approximations, find an expression for the electrostatic repulsion in terms of A and r_0.

(f) Combine the binding and repulsive expressions to get an approximation of nuclear binding energy in terms of the binding energy per pair, ϵ_b, the atomic mass, A, and the charge radius of the hydrogen nucleus, r_0. Create a plot of the binding energy as a function of atomic mass for $150 < A < 200$. Take $\epsilon_b = 1.2$ MeV, the binding energy of the deuterium nucleus, ${}^2_1$H.

(g) More informative is a plot of E_b/A as a function of A, as it affords a clearer way to compare the degree of binding among nuclei. Create such a plot over the same atomic mass range.

(h) The derived expression is (part of) an important result from the liquid drop model, known as the semi-empirical (or Weizsäcker, or Bethe–Weizsäcker) mass formula, which predicts nuclear masses:

$$m(Z, N) = Zm_p + Nm_n - E_b$$

in natural units (electron binding energy is small and ignored here). Compute the mass of each of the following atoms using the (partial) semi-empirical mass formula just derived and compare with measured masses.

i. ${}^{56}_{26}$Fe
ii. ${}^{62}_{28}$Ni
iii. ${}^{85}_{37}$Rb
iv. ${}^{107}_{47}$Ag
v. ${}^{195}_{78}$Pt
vi. ${}^{197}_{79}$Au

The full semi-empirical mass formula contains five terms: volume energy, surface energy, Coulomb energy, asysmmetry (or Pauli) energy (due to the Pauli exclusion principle), and pairing energy (due to spin coupling, pairing is more stable than unpaired nucleons). The derivation includes only the first three terms. The asymmetry and pairing terms are the only terms to reflect the quantum nature of the nucleus. Furthermore, the full formula ignores the internal nuclear shell structure of the nucleus, which is more important for small nuclei than for large ones.

(i) The derivative with respect to mass number of the binding energy indicates whether or not a nucleus is stable against alpha transformation. Basically, if the difference between the binding energies per nucleon of the parent and daughter nuclei is less than the binding energy per nucleon of the helium-4 nucleus, the parent will transform into the daughter, emitting an alpha particle. Conversely, if

$$\frac{E_b({}^A_Z\mathrm{P}) - E_b({}^{A-4}_{Z-2}\mathrm{D})}{4} \approx \frac{dE_b}{dA} > \frac{E_b}{A}({}^4_2\mathrm{He})$$

then the parent is stable against an alpha transformation. Plot the derivative of the binding energy with respect to mass number as a function of mass number for $150 < A < 200$ and determine the mass number A above which nuclei are not stable against alpha transformation. That is, find the value of A on the plot when

$$\frac{dE_b}{dA} < \frac{E_b}{A}(^4_2\mathrm{He}).$$

$\frac{E_b}{A}(^4_2\mathrm{He})$ was calculated in exercise set 4.

2. **$^{20}_{12}$Mg, $^{20}_{11}$Na, $^{20}_{10}$Ne, $^{20}_{9}$F, $^{20}_{8}$O, $^{20}_{7}$N, and $^{20}_{6}$C are isobars, nuclides with the same atomic mass number A but different numbers of protons Z and neutrons N.**
   (a) **Compute binding energies per nucleon for each of these isobars and identify the most stable nucleus.**
   (b) **How close is the estimate of the semi-empirical mass formula to these values? Discuss possible sources of any discrepancy.**

The liquid drop model averages nuclear effects without considering nucleons individually. It can therefore describe, for example, the average binding energy per nucleon of many nuclei, but it cannot describe excited energy states or nuclear magnetic moments. To describe these requires a microscopic model of individual nucleons.

Experiments found that nuclei in which Z and/or N equals 2, 8, 20, 28, 50, 82, or 126 are both abundant and particularly stable. These numbers have become known as "magic numbers." Their existence implies a shell structure, especially since the last or magic nucleon that "completes a shell" has an especially high binding energy, and the energy of its first excited state is larger than the first excited state of a nearby nucleus without any magic numbers. All of this suggests a nuclear shell model of nuclei, much like the atomic shell model.

The atomic shell model of a nucleus of Z protons first assumes Z electrons, moving independently in an average nuclear Coulomb field, fill successive energy levels. Small corrections are then introduced to account for effects due to electron-electron and electron-nucleus interactions.

A very approximate nuclear shell model assumes nucleons move independently in three dimensions under the influence of an average harmonic oscillator potential,

$$V = \frac{1}{2}kR^2 = \frac{1}{2}m\omega^2 R^2$$

where k is the "spring constant" (indicating the strength of the interaction), m is the nucleon mass, and ω is the nucleon's angular velocity. Solved quantum mechanically, this potential leads to energy levels

$$E = (2k + \ell + \frac{3}{2})\hbar\omega$$

Here, k is a non-negative integer (including zero) which identifies the order of the generalized Laguerre polynomial that is the radial part of the solution to the Schrödinger equation for the three-dimensional harmonic oscillator potential, and ℓ, the orbital angular momentum variable ($|\vec{l}| = \sqrt{\ell(\ell+1)}\hbar$), can be any non-negative integer (including zero). The energy quantum number is defined to be $\mathcal{N} \equiv 2k + \ell$, which is thus also a non-negative integer (including zero). Unlike the atomic case, ℓ is not limited by $\mathcal{N}$, but, as can be seen from the definition, when ℓ is even or zero, so is $\mathcal{N}$, and when ℓ is odd, so is $\mathcal{N}$.

As is the case for atomic states, spectroscopic notation has been standardized to indicate nuclear orbital angular momentum states:

l:	0	1	2	3	4	5	$\cdots$
Symbol:	s	p	d	f	g	h	$\cdots$

The energy order of an ℓ-state is given by prefixing the symbol with the value of $n = k + 1$, which is the number of nodes in the generalized Laguerre polynomial. Thus, $1s$ is the lowest energy zero orbital angular momentum ($\ell = 0$) state, and $1p$ is the lowest energy $\ell = 1$ state, while $2d$ is the next-to-lowest energy $\ell = 2$ state.

Nucleons have intrinsic angular momentum (spin); they are spin-1/2 objects, and so there are two states for each orbital angular momentum level, a total of $4\ell + 2$ states: 2 ($\ell = 0$) s states, 6 ($\ell = 1$) p states, 10 ($\ell = 2$) d states, 14 ($\ell = 3$) f states, 28 ($\ell = 4$) g states, and so forth, for every energy level n.

The two angular momenta, orbital and spin, combine in a spin-orbit interaction: the energy levels split depending on the total angular momentum, $j = \ell + s$ or $j = \ell - s$. When $\vec{\ell}$ and $\vec{s}$ are parallel, that is, when $j = \ell + \frac{1}{2}$, the interaction energy is positive. When $\vec{\ell}$ and $\vec{s}$ are anti-parallel, that is, when $j = \ell - \frac{1}{2}$, the interaction energy is negative.

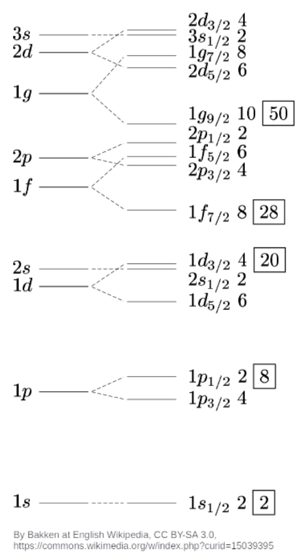
$$\begin{aligned} \vec{\ell} \cdot \vec{s} &= \frac{1}{2}[j(j+1) - \ell(\ell+1) - s(s+1)]\hbar^2 \\ &= \begin{cases} \frac{\ell}{2}\hbar^2 & j = \ell + \frac{1}{2} \\ -\frac{\ell+1}{2}\hbar^2 & j = \ell - \frac{1}{2} \end{cases} \end{aligned}$$

As a result, orbital momentum states are split into two sets of $2j + 1$ states. Thus, for example, the 10 ($\ell = 2$) d states are split into 4 $j = \frac{3}{2}$ states (in which the orbital and spin momenta are anti-parallel) and 6 $j = \frac{5}{2}$ states (in which the orbital and spin momenta are parallel).

Splitting due to spin-orbit interactions is much more pronounced in nuclei than it is among atomic electron energy levels. Further, in nuclei, the $j = \ell + \frac{1}{2}$ orbit (in which the orbital and spin momenta are parallel) has lower energy than the $j = \ell - \frac{1}{2}$ orbit (in which the orbital and spin momenta are anti-parallel), the opposite of what happens in atoms. Notice, too, that the splitting separation increases with ℓ.

The orbits occupied by nuclei are then designated by postpending the total angular momentum j to the nuclear spectroscopic notation as a subscript. For example, a nucleon in one of the 8 lowest energy $\ell = 3$ orbits, that is, with $n = 0$, $\ell = 3$, $j = \ell + \frac{1}{2} = \frac{7}{2}$, is designated $1f_{7/2}$.

The Pauli exclusion principle, as applied to nucleons–no two of them can have the same set of energy (n), orbital angular momentum (ℓ), total angular momentum (j), and z-component of the total angular momentum quantum numbers (m_j)–then yields the energy spectra including the magic numbers.



As with atoms, states with even orbial angular momentum ℓ are even (positive) parity states, while odd ℓ yields odd (negative) parity states.

Pairs of protons and of neutrons tend to align so as to produce states of zero total angular momentum. The result is that even-even nuclei (two protons and two neutrons per level) have zero total angular momentum. Even-odd or odd-even nuclei have the total angular momentum of the last (odd) nucleon. Odd-odd nuclei are complicated.

3. What are the total angular momenta for the following $^{13}_{6}C$ states?

(a) The ground state (all protons and neutron levels are filled through $1p_{3/2}$, and the extra neutron is in the $1p_{1/2}$ level).

(b) An excited state in which everything is the same as (a) except that the extra neutron is in the $2s_{1/2}$ level.

(c) An excited state in which everything is the same as (a) except that the extra neutron is in the $1d_{5/2}$ level.

(d) **An excited state in which all proton levels are filled through $1p_{3/2}$, two neutrons are in the $1s_{1/2}$ level, three neutrons are in the $1p_{3/2}$ level, and two neutrons are in the $1p_{1/2}$ level.**

4. **Find, by referring to the table, the ground-state total angular momentum of**

   (a) $^{15}_{8}\mathrm{O}$
   (b) $^{20}_{10}\mathrm{Ne}$
   (c) $^{39}_{19}\mathrm{K}$
   (d) $^{41}_{20}\mathrm{Ca}$
   (e) $^{80}_{36}\mathrm{Kr}$
   (f) $^{91}_{40}\mathrm{Zr}$

5. **What possible ground-state angular momentum values could $^{32}_{15}\mathrm{P}$ have?**